

AMENABILITY AND SUBEXPONENTIAL SPECTRAL GROWTH RATE OF DIRICHLET FORMS ON VON NEUMANN ALGEBRAS

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ABSTRACT. In this work we apply Noncommutative Potential Theory to prove (relative) amenability and the (relative) Haagerup Property (H) of von Neumann algebras in terms of the spectral growth of Dirichlet forms. Examples deal with (inclusions of) countable discrete groups and free orthogonal compact quantum groups.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS.

Classical results relate the metric properties of conditionally negative definite functions on a countable discrete group Γ to its approximation properties. For example, there exists a *proper*, conditionally negative definite function ℓ on Γ if and only if there exists a sequence $\varphi_n \in c_0(\Gamma)$ of normalized, positive definite functions, vanishing at infinity and converging pointwise to the constant function 1.

In a celebrated work [Haa2], U. Haagerup proved that the length function of a free group \mathbb{F}_n with $n \in \{2, \dots, \infty\}$ generators is negative definite, thus establishing for free groups the above approximation property. Since then the property is referred to as Haagerup Approximation Property (H) or Gromov a-T-menability (see [CCJJV]).

In addition, if for a conditionally negative definite function ℓ on a countable discrete group Γ , the series $\sum_{g \in \Gamma} e^{-t\ell(g)}$ converges for all $t > 0$, then there exists a sequence $\varphi_n \in l^2(\Gamma)$ of normalized, positive definite functions, converging pointwise to the constant function 1 ([GK Thm 5.3]). This latter property is just one of the several equivalent appearances of *amenability*, a property introduced by J. von Neumann in 1929 [vN] in order explain the Banach-Tarski paradox in Euclidean spaces \mathbb{R}^n exactly when $n \geq 3$.

In this note we are going to discuss extensions of the above results concerning amenability for σ -finite von Neumann algebras N .

The direction along which we are going to look for substitutes of the above summability condition related to amenability, is that of Noncommutative Potential Theory.

This is suggested by a recent result by Caspers-Skalski [CaSk] asserting that N has the (suitably formulated) Haagerup Approximation Property (H) if and only if there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the standard Hilbert space $L^2(N)$, having *discrete spectrum*.

The link between the properness condition for a conditionally negative definite function ℓ on a countable discrete groups Γ and the generalized one on von Neumann algebras, relies in the fact that, when the von Neumann algebra $N = L(\Gamma)$ is the one generated by the left regular representation of Γ , the quadratic form $\mathcal{E}_\ell[a] = \sum_{g \in \Gamma} \ell(g)|a(g)|^2$ on the standard space $L^2(L(\Gamma), \tau) \simeq l^2(\Gamma)$ is a Dirichlet form if and only if the function ℓ is conditionally negative

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definite and its spectrum is discrete if and only if ℓ is proper. Moreover, on a countable, finitely generated, discrete group Γ with polynomial growth, there exist a conditionally negative definite functions ℓ , having polynomial growth and growth dimensions arbitrarily close to the homogeneous dimension of Γ (see [CS5]).

This point of view thus suggests that a condition providing amenability of a von Neumann algebra with faithful normal state (N, ω) could be the *subexponential spectral growth* of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the standard space $L^2(N, \omega)$, i.e. the summability of the series $\sum_{k \geq 0} e^{-t\lambda_k}$ for all $t > 0$, where $\lambda_0, \lambda_1, \dots$ are the eigenvalues of $(\mathcal{E}, \mathcal{F})$.

The second fundamental fact that will allow to use Dirichlet forms to investigate the amenability of a von Neumann algebra, is the possibility to express this property in terms of Connes' *correspondences*: N is amenable if and only if the identity or standard N - N -correspondence $L^2(N)$ is weakly contained in the coarse or Hilbert-Schmidt N - N -correspondence $L^2(N) \otimes L^2(N)$ (see [Po1]).

In the second part of the work we provide a condition guaranteeing the *relative amenability* of an inclusion $B \subseteq N$ of finite von Neumann algebras introduced by Popa [Po1,2], in terms of the existence of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N)$ having *relative subexponential spectral growth*. Also this result is based on the possibility to express the relative amenability of a von Neumann algebra N with respect to a subalgebra $B \subseteq N$ in terms of the weak containment of the identity correspondence $L^2(N)$ in the relative tensor product correspondence $L^2(N) \otimes_B L^2(N)$ introduced by Sauvageot [S1], [Po2].

Using a suitable Dirichlet form constructed in [CFK], we apply the above result to prove amenability of the von Neumann algebra of the free orthogonal quantum group O_2^+ and Haagerup Property (H) of the free orthogonal quantum groups O_N^+ for $N \geq 3$, results obtained by M. Brannan [Bra] with completely different methods.

A detailed discussion of the relative Haagerup Property (H) inclusions of countable discrete groups in terms of conditionally negative definite functions is presented.

The paper is organized as follows: in Section 2 we provide the necessary tools on noncommutative potential theory on von Neumann algebra as Dirichlet forms, Markovian semigroups and resolvents.

Section 3 is dedicated to a brief presentation of constructions of Dirichlet forms in several settings as well as to recall the connection with the first order differential calculus, in the trace case.

In Section 4 we first recall some equivalent construction of the coarse or Hilbert-Schmidt correspondence of a von Neumann algebra N and some connection between the modular theories of N , of its opposite N^o , and of their spatial tensor product $N \overline{\otimes} N^o$. Then we introduce the spectral growth rate of a Dirichlet form and we prove the first main result of the work about the amenability of von Neumann algebra admitting a Dirichlet form with subexponential spectral growth rate. This part terminates with an application to the amenability of countable discrete groups and with a new and independent proof of a result of M. Brannan [Bra] about the amenability of the free orthogonal quantum group O_2^+ .

Section 5 starts recalling some fundamental tool of the basic construction $\langle N, B \rangle$ for inclusions $B \subseteq N$ of finite von Neumann algebras, needed to prove the second main result of the work concerning the amenability of N with respect to its subalgebra B . To formulate the criterion, we introduce the spectral growth rate of a B -invariant Dirichlet form on the standard space $L^2(N)$ relatively to the subalgebra B , using the compact ideal space $\mathcal{J}(\langle N, B \rangle)$ of $\langle N, B \rangle$ (cf. [PO1,2]). The section terminates discussing relative amenability for two natural subalgebras

$B_{\min} \subseteq N$ and $B_{\max} \subseteq N$ associated to a Dirichlet form, especially in the countable, discrete group setting.

In Section 6 we extend the spectral characterization of the Haagerup Property (H) of von Neumann algebras with countable decomposable center due to M. Caspers and A. Skalski [CaSk] to the Relative Haagerup Property (H) for inclusions of finite von Neumann algebras $B \subseteq N$ formulated by S. Popa [Po 1,2].

In Section 7 we discuss the relative Haagerup Property (H) for inclusions $H < G$ of countable discrete groups in terms of the existence of an H -invariant conditionally negative definite function on G which is proper on the homogeneous space G/H and in terms of quasi-normality of H in G .

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2. DIRICHLET FORMS ON σ -FINITE VON NEUMANN ALGEBRAS

Recall that a von Neumann algebra N is σ -finite, or countably decomposable, if any collection of mutually orthogonal projections is at most countable and that this property is equivalent to the existence of a normal, faithful state. This is the case, for example, if N acts faithfully on a separable Hilbert space.

Let us consider on a σ -finite von Neumann algebra N a fixed faithful, normal state $\omega \in N_{*+}$. Let us denote by $(N, L^2(N, \omega), L_+^2(N, \omega), J_\omega)$ the standard form of N and by $\xi_\omega \in L_+^2(N, \omega)$ the cyclic vector representing the state (see [Haa1]).

For a real vector $\xi = J_\omega \xi \in L^2(N, \omega)$, let us denote by $\xi \wedge \xi_\omega$ the Hilbert projection of the vector ξ onto the closed and convex set $C_\omega := \{\eta \in L^2(N, \omega) : \eta = J_\omega \eta, \xi_\omega - \eta \in L_+^2(N, \omega)\}$.

We recall here the definition of Dirichlet form and Markovian semigroup (see [C1]) on a generic standard form of a σ -finite von Neumann algebra. For a definition particularized to the Haagerup standard form see [GL].

Definition 2.1 (Dirichlet forms on von Neumann algebras). A densely defined, nonnegative and lower semicontinuous quadratic form $\mathcal{E} : L^2(N, \omega) \rightarrow [0, +\infty]$ is said to be:

i) *real* if

$$(2.1) \quad \mathcal{E}[J_\omega(\xi)] = \mathcal{E}[\xi] \quad \xi \in L^2(N, \omega);$$

ii) a *Dirichlet form* if it is real and *Markovian* in the sense that

$$(2.2) \quad \mathcal{E}[\xi \wedge \xi_\omega] \leq \mathcal{E}[\xi] \quad \xi = J_\omega \xi \in L^2(N, \omega);$$

iii) a *completely Dirichlet form* if all the canonical extensions \mathcal{E}_n to $L^2(\mathbb{M}_n(N), \omega \otimes \text{tr}_n)$

$$(2.3) \quad \mathcal{E}_n[[\xi_{i,j}]_{i,j=1}^n] := \sum_{i,j=1}^n \mathcal{E}[\xi_{i,j}] \quad [\xi_{i,j}]_{i,j=1}^n \in L^2(\mathbb{M}_n(N), \omega \otimes \text{tr}_n),$$

are Dirichlet forms.

By the self-polarity of the standard cone $L_+^2(N, \omega)$, any real vector $\xi = J_\omega \xi \in L^2(N, \omega)$ decomposes uniquely as a difference $\xi = \xi_+ - \xi_-$ of two positive, orthogonal vectors $\xi_\pm \in L_+^2(N, \omega)$ (the positive part ξ_+ being just the Hilbert projection of ξ onto the positive cone). The modulus of ξ is then defined as the sum of the positive and negative parts $|\xi| := \xi_+ + \xi_-$.

Notice that, in general, the contraction property

$$\mathcal{E}[|\xi|] \leq \mathcal{E}[\xi] \quad \xi = J_\omega \xi \in L^2(A, \omega)$$

is a consequence of Markovianity and that it is actually equivalent to it when $\mathcal{E}[\xi_\omega] = 0$.

The domain of the Dirichlet form is defined as the (dense) subspace of $L^2(N, \omega)$ where the quadratic form is finite: $\mathcal{F} := \{\xi \in L^2(N, \omega) : \mathcal{E}[\xi] < +\infty\}$. We will denote by $(L, D(L))$ the densely defined, self-adjoint, lower semibounded operator on $L^2(A, \tau)$ associated with the closed quadratic form $(\mathcal{E}, \mathcal{F})$

$$\mathcal{F} = D(\sqrt{L}) \quad \text{and} \quad \mathcal{E}[\xi] = \|\sqrt{L}\xi\|^2 \quad \xi \in D(\sqrt{L}) = \mathcal{F}.$$

Definition 2.2 (Markovian semigroups on standard forms of von Neumann algebras).

- a) A bounded operator T on $L^2(N, \omega)$ is said to be
- i) *real* if it commutes with the modular conjugation: $TJ_\omega = J_\omega T$,
- ii) *positive* if it leaves globally invariant the positive cone: $T(L_+^2(N, \omega)) \subseteq L_+^2(N, \omega)$,
- iii) *Markovian* if it is real and it leaves globally invariant the closed, convex set C_ω :

$$T(C_\omega) \subseteq C_\omega,$$

- iv) *completely positive, resp. completely Markovian*, if it is real and all of its matrix amplifications $T^{(n)}$ to $L^2(\mathbb{M}_n(N), \omega \otimes \text{tr}_n) \simeq L^2(N, \tau) \otimes L^2(\mathbb{M}_n(\mathbb{C}), \text{tr}_n)$

$$T^{(n)}[[\xi_{i,j}]_{i,j=1}^n] := \sum_{i,j=1}^n [T\xi_{i,j}]_{i,j=1}^n \quad [\xi_{i,j}]_{i,j=1}^n \in L^2(\mathbb{M}_n(N), \omega \otimes \text{tr}_n),$$

are positive, resp. Markovian;

- b) A strongly continuous, uniformly bounded, self-adjoint semigroup $\{T_t : t > 0\}$ on $L^2(N, \omega)$ is said to be real (resp. positive, Markovian, completely positive, completely Markovian) if the operators T_t are real (resp. positive, Markovian, completely positive, completely Markovian) for all $t > 0$.

Notice that if N is abelian, then positive (resp. Markovian) operators are automatically completely positive (resp. completely Markovian).

Theorem 2.3. (*Generalized Beurling-Deny correspondence [C1]*). *Dirichlet forms are in one-to-one correspondence with Markovian semigroups through the relations*

$$T_t = e^{-tL} \quad t \geq 0$$

where $(L, D(L))$ is the self-adjoint operator associated to the quadratic form $(\mathcal{E}, \mathcal{F})$.

Dirichlet forms and Markovian semigroups are also in correspondence with a class of semigroups on the von Neumann algebra. To state this fundamental relation, let us consider the *symmetric embedding* i_ω determined by the cyclic vector ξ_ω

$$i_\omega : N \rightarrow L^2(N, \omega) \quad i_\omega(x) := \Delta_\omega^{\frac{1}{4}} x \xi_\omega \quad x \in N.$$

Here, Δ_ω is the modular operator associated with the faithful state ω [T]. We will denote by $\{\sigma_t^\omega : t \in \mathbb{R}\}$ the modular automorphisms group associated to ω and by $N_{\sigma^\omega} \subseteq N$ the subalgebra of elements which are analytic with respect to it.

Theorem 2.4. ([C1]) *(Completely) Dirichlet forms $(\mathcal{E}, \mathcal{F})$ and (completely) Markovian semigroups $\{T_t : t > 0\}$ on $L^2(N, \omega)$ are in one-to-one correspondence with those C_0^* -continuous,*

(completely) positive and contractive semigroups $\{S_t : t > 0\}$ on the von Neumann algebra N which are modular ω -symmetric in the sense that

$$(2.4) \quad \omega(S_t(x)\sigma_{-i/2}^\omega(y)) = \omega(\sigma_{-i/2}^\omega(x)S_t(y)) \quad x, y \in N_{\sigma^\omega}, \quad t > 0,$$

through the relation $i_\omega(S_t(x)) = T_t(i_\omega(x)) \quad x \in N, \quad t > 0.$

Relation (2.4) is called modular symmetry and it is equivalent to

$$(2.5) \quad (J_\omega y \xi_\omega | S_t(x) \xi_\omega) = (J_\omega S_t(y) \xi_\omega | x \xi_\omega) \quad x, y \in N, \quad t > 0.$$

Remark 2.5. i) In case ω is a trace, the symmetric embedding reduces to $i_\omega(x) = x\xi_\omega$ while the modular symmetry simplifies to $\omega(S_t(x)y) = \omega(xS_t(y))$ for $x, y \in N$ and $t > 0$.

ii) In this work we only deal with aspects of noncommutative potential theory on σ -finite algebras and with respect to a faithful normal state. The theory of Dirichlet forms and Markovian semigroups on operator algebras was initially formulated by L. Gross [G1,2] and S. Albeverio and R. Hoegh-Krohn [AHK] on semifinite von Neumann algebras with respect to lower semicontinuous, normal, faithful, semifinite traces. The framework was then extended to general von Neumann algebras with reference to a fixed faithful, normal state [GL1], [C1] and later with respect to a lower semicontinuous, normal, faithful weight by J.M. Lindsay and S. Goldstein [GL2].

*To shorten notations, in the forthcoming part of the paper
"Dirichlet form" will always mean "completely Dirichlet form" and
"Markovian semigroup" will always mean "completely Markovian semigroup".*

Whenever no confusion can arise, the modular conjugation will be sometime denoted by J in place of J_ω .

3. EXAMPLES OF DIRICHLET FORMS

To familiarize with the notions introduced so far, we present in this section some examples of various origins. One may consult the fundamental works [BeDe], [FOT] for the commutative case and [C2], [C3] for surveys in the noncommutative setting.

3.1. Dirichlet spaces on commutative von Neumann algebras. a) The archetypical Dirichlet form on the Euclidean space \mathbb{R}^n or, more generally, on any Riemannian manifold V , endowed with its Riemannian measure m , is the Dirichlet integral

$$\mathcal{E}[a] = \int_V |\nabla a|^2 dm \quad a \in L^2(V, m).$$

In this case the trace on $L^\infty(V, m)$ is given by the integral with respect to the measure m and the Dirichlet space is the Sobolev space $H^1(V) \subset L^2(V, m)$. The associated Markovian semigroup is the familiar heat semigroup of the Riemannian manifold.

Interesting variations of the above Dirichlet integral are the Dirichlet forms of type

$$\mathcal{E}[a] := \int_{\mathbb{R}^n} |\nabla a|^2 d\mu \quad a \in L^2(V, \mu).$$

For suitable choices of positive Radon measures μ on \mathbb{R}^n , they are ground state representations of Hamiltonian operators in Quantum Mechanics.

b) Dirichlet forms are a fundamental tool to introduce differential calculus and study Markovian stochastic processes on fractal sets (see [Ki], [CS3], [CGIS 1,2]). To recall just one of the most studied cases, we consider the Sierpinski gasket.

Given the vertices $\{p_1, p_2, p_3\}$ of an equilateral triangle in \mathbb{R}^2 , consider the contractions $F_i(x) := (x + p_i)/2$ defined for $x \in \mathbb{R}^2$ and $i = 1, 2, 3$. The Sierpinski gasket K is then defined as the unique fixed point of the contraction map $C \mapsto F_1(C) \cup F_2(C) \cup F_3(C)$ on the space of compact subsets of \mathbb{R}^2 , endowed with the Hausdorff metric. The defining property $K = F_1(K) \cup F_2(K) \cup F_3(K)$, called *self-similarity*, is responsible for several *singular* behaviours: K is not a manifold and it does not admit a universal cover.

Consider on K a self-similar volume measure μ defined, for some fixed $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\sum_{i=1}^3 \alpha_i = 1$, as the unique solution of the self-similarity equation

$$\int_K f d\mu = \sum_{i=1}^3 \alpha_i \int_K (f \circ F_i) d\mu \quad f \in C(K).$$

To construct a class of Dirichlet forms K , let us fix the following notations:

- word spaces: $\sum_0 := \emptyset$, $\sum_m := \{1, 2, 3\}^m$, $\sum := \bigcup_{m \geq 0} \sum_m$
- length of a word $\sigma \in \sum_m$: $|\sigma| := m$
- iterated contractions: $F_\sigma := F_{i_{|\sigma|}} \circ \dots \circ F_{i_1}$ if $\sigma = (i_1, \dots, i_{|\sigma|})$
- vertices sets: $V_0 := \{p_1, p_2, p_3\}$, $V_m := \bigcup_{|\sigma|=m} F_\sigma(V_0)$.

The quadratic form $\mathcal{E}_0 : L^\infty(V_0) \rightarrow [0, +\infty)$ on the three-point set V_0 defined by

$$\mathcal{E}_0[a] := (a(p_1) - a(p_2))^2 + (a(p_2) - a(p_3))^2 + (a(p_3) - a(p_1))^2,$$

is a Dirichlet form with respect to any measure on V_0 . It is the first of a sequence of Dirichlet forms \mathcal{E}_m defined on vertices sets V_m defined by

$$\mathcal{E}^m[a] := \sum_{|\sigma|=m} \left(\frac{5}{3}\right)^m \mathcal{E}_0[a \circ F_\sigma] \quad a \in L^\infty(V_m)$$

with respect to any measure on V_m . The quadratic form $\mathcal{E}_K : L^\infty(K, \mu) \rightarrow [0, +\infty]$ defined by

$$\mathcal{E}_K[a] := \lim_{m \rightarrow +\infty} \mathcal{E}^m[a|_{V_m}].$$

is a Dirichlet form which is self-similar in the sense that

$$\mathcal{E}_K[a] = \frac{5}{3} \sum_{i=1}^3 \mathcal{E}_K[a \circ F_i] \quad a \in C(K).$$

Among the singular behaviors of this Dirichlet form compared to the Dirichlet integral of a Riemannian, we recall the fact that the energy distribution or *carré du champ* associated to \mathcal{E}_K is singular with respect to any of the self-similar measures above: in other words energy and volume are distributed singularly.

3.2. Dirichlet forms on the Clifford C^* -algebra of a Riemannian manifold. On a Riemannian manifold (V, g) consider the associated Clifford bundle $\mathcal{C}\ell(V)$, whose fiber $\mathcal{C}\ell_x(V)$ at $x \in V$ is the (complexification of the) Clifford algebra of the tangent space $T_x V$. By Clifford multiplication, $\mathcal{C}\ell_x(V)$ is canonically a (finite dimensional, noncommutative) C^* -algebra with a canonical trace τ_x . More precisely, $\mathcal{C}\ell_x(V)$ is the full matrix algebra $M_{2^m}(\mathbb{C})$ if n is even, $n = 2m$, and it is the sum of two full matrix algebras $M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$ if n is odd, $n = 2m+1$. Sections of $\mathcal{C}\ell(V)$ vanishing at infinity thus form, in a natural way, a C^* -algebra $C_0^*(V)$, called the *Clifford C^* -algebra* of V . Its center reduces to the commutative C^* -subalgebra $C_0(V)$ of continuous functions vanishing at infinity on V if n is even and it reduces to $C_0(V) \oplus C_0(V)\nu$ if n is odd, ν being the volume form. The Riemannian measure m on (V, g) gives rise to a

trace $\tau = \int_V \tau_x m(dx)$ on $C_0^*(V)$ whose associated GNS space $L^2(C_0^*(V), \tau)$ coincides with the Hilbert space $L^2(\mathcal{C}\ell(V))$ of square integrable sections of the Clifford bundle. The Levi-Civita connection of (V, g) , through its covariant derivative ∇ , gives rise to a Dirichlet form on $C_0^*(V)$

$$\mathcal{E}_B[\sigma] := \int_V |\nabla \sigma|^2,$$

whose associated self-adjoint operator is the Bochner Laplacian $\Delta_B = \nabla^* \nabla$ of V ([DR1],[DR2]). The Dirac operator D on $\mathcal{C}\ell(V)$ gives rise to a nonnegative, closed quadratic form

$$\mathcal{E}_D[\sigma] := \|D\sigma\|^2,$$

whose associated self-adjoint operator is the so called Dirac Laplacian $\Delta_D = D^2$. Under the canonical isomorphism of Hilbert spaces between $L^2(\mathcal{C}\ell(V))$ and the Hilbert space $L^2(\Lambda^*V)$ of square integrable sections of the exterior bundle of V , the Dirac Laplacian translates into the Hodge-de Rham Laplacian $\Delta_D \simeq (d + d^*)^2 = d^*d + dd^*$, where d is the exterior derivative and d^* its adjoint. Manifold in which \mathcal{E}_D is a Dirichlet form, i.e. the heat semigroup $e^{-t\Delta_D}$ is a Markovian semigroup on the Clifford C^* -algebra $C_0^*(V)$, have been characterized geometrically as those in which the curvature operator \mathcal{R}_V of V is nonnegative (see [CS3]).

3.3. Dirichlet spaces on group von Neumann algebras. Let Γ be a countable discrete group, with unit $e \in \Gamma$, whose elements will be denoted by s, t, \dots . Denote by λ its left regular representation on $l^2(\Gamma)$ acting by

$$(\lambda(s)a)(t) := a(s^{-1}t) \quad s, t \in \Gamma, \quad a \in l^2(\Gamma)$$

and by $L(\Gamma)$ its left von Neumann algebra in $\mathcal{B}(l^2(\Gamma))$ generated by the unitaries $\{\lambda(s) \in \mathcal{B}(l^2(\Gamma)) : s \in \Gamma\}$ (see [Dix]). By a von Neumann theorem, $L(\Gamma)$ coincides with the double commutant of the above set of unitaries

$$L(\Gamma) = \{\lambda(s) \in \mathcal{B}(l^2(\Gamma)) : s \in \Gamma\}''$$

It results that

$$\sqrt{\sum_{s \in \Gamma} |x(s)|^2} \leq \|x\|_{L(\Gamma)} \leq \sum_{s \in \Gamma} |x(s)| \leq +\infty$$

so that one has the contractive embeddings $l^1(\Gamma) \subseteq L(\Gamma) \subseteq l^2(\Gamma)$ with dense images. In terms of the coefficients, the product in $L(\Gamma)$ reduces to a convolution

$$(x * y)(s) := \sum_{t \in \Gamma} x(t)y(t^{-1}s) \quad s \in \Gamma, x, y \in L(\Gamma)$$

while the involution is given by $(x^*)(s) := \overline{x(s^{-1})}$, $s \in \Gamma$. The left regular representation of Γ extends to a $*$ -representation of the von Neumann algebra and it will be denoted by the same symbol. The functional $L(\Gamma) \ni x \mapsto x(e) \in \mathbb{C}$ is a tracial state $\tau : L(\Gamma) \rightarrow \mathbb{C}$ and the associated G.N.S. representation coincides with the left regular representation above. In particular the G.N.S. Hilbert space $L^2(L(\Gamma), \tau)$ can be identified as complex Hilbert space with $l^2(\Gamma)$ and its positive standard cone $L_+^2(L(\Gamma), \tau)$ with the cone of square integrable functions $\xi \in l^2(\Gamma)$ which are positive definite in sense that

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j \xi(s_i^{-1} s_j) \geq 0 \quad n \in \mathbb{N}^*, \quad s_1, \dots, s_n \in \Gamma, \quad \alpha_1, \dots, \alpha_n \in \mathbb{C}.$$

Parallel arguments, starting from the right representation ρ of Γ on $l^2(\Gamma)$ given by

$$(\rho(s)a)(t) := a(ts) \quad s, t \in \Gamma, \quad a \in l^2(\Gamma),$$

provide the right von Neumann algebra represented in $l^2(\Gamma)$ which coincides with the commutant of the left one: $L(\Gamma)' = R(\Gamma)$.

Conditionally negative definite (c.n.d.) functions $\ell : \Gamma \rightarrow [0, +\infty)$ (see [Boz], [CCJJV]), characterized by the property

$$\sum_{i,j=1}^n \alpha_i = 0 \quad \Rightarrow \quad \sum_{i,j=1}^n \overline{\alpha_i} \alpha_j \xi(s_i^{-1} s_j) \leq 0 \quad n \in \mathbb{N}^*, \quad s_1, \dots, s_n \in \Gamma, \quad \alpha_1, \dots, \alpha_n \in \mathbb{C},$$

give rise to Dirichlet forms

$$\mathcal{E}_\ell[\xi] := \sum_{s \in \Gamma} |\xi(s)|^2 \ell(s),$$

defined on the subspace \mathcal{F}_ℓ of $l^2(\Gamma)$ where the series is finite (see [CS1], [C2]). The associated Markovian semigroup is simply given by a multiplication operator

$$T_t(a)(s) = e^{-t\ell(s)} a(s) \quad t > 0, \quad s \in G, \quad a \in l^2(\Gamma).$$

The positive preserving property of the semigroup can be directly verified just recalling the fact that the negative exponential of a c.n.d. function is a positive definite one and that the pointwise product of positive definite functions is positive definite too.

Examples of the above framework arise on \mathbb{Z}^n , where as negative definite function one chooses the square of Euclidean length $\ell(k) := |k|^2$. In this case the abelian discrete group \mathbb{Z}^n is the dual of the compact abelian torus \mathbb{T}^n and one recognizes the Dirichlet form \mathcal{E}_ℓ associated to the c.n.d. function $\ell(k) := |k|^2$, as the Fourier transform of the Dirichlet integral on the flat \mathbb{T}^n . The choice of the Euclidean length $\ell(k) := |k|$ provides a Dirichlet form \mathcal{E}_ℓ whose generator L corresponds, under Fourier transform, to the spectral square root of the Laplace operator on the flat \mathbb{T}^n .

On the free group \mathbb{F}_n with n generators $S := \{a_1, \dots, a_n\}$, the most important c.n.d functions are the word length functions ℓ_S associated to the system S (see [Haa2] [deH]): for $w \in \mathbb{F}_n$, $\ell_S(w)$ is defined as the smallest integer k for which there exists a sequence $s_1, \dots, s_k \in S \cup S^{-1}$ such that $w = s_1 \cdots s_k$.

3.4. Dirichlet forms on noncommutative tori. Noncommutative tori are a family of C^* -algebras which represent a sort of gymnasium for Noncommutative Geometry [Co2]. They are defined, for any fixed irrational $\theta \in [0, 1]$, as the universal C^* -algebras A_θ generated by two unitaries U and V , satisfying the relation

$$VU = e^{2i\pi\theta} UV.$$

The functional $\tau : A_\theta \rightarrow \mathbb{C}$ characterized by

$$\tau(U^n V^m) = \delta_{n,0} \delta_{m,0} \quad n, m \in \mathbb{Z}$$

is a faithful, tracial state. The G.N.S. representation π_θ of A_θ with respect to the trace can be realized on $l^2(\mathbb{Z}^2)$ as follows

$$(\pi_\theta(U)\xi)(z_1, z_2) = \xi(z_1 - 1, z_2), \quad (\pi_\theta(V)\xi)(z_1, z_2) = e^{2\pi i \theta z_1} \xi(z_1, z_2 - 1), \quad \xi \in l^2(\mathbb{Z}^2)$$

and the cyclic vector representing the trace is $\delta_{(0,0)} \in l^2(\mathbb{Z}^2)$. We will denote by N_θ the von Neumann algebra acting in $l^2(\mathbb{Z}^2)$ which is generated by the unitaries $u := \pi_\theta(U)$, $v := \pi_\theta(V)$. The *heat semigroup* $\{T_t : t \geq 0\}$ on N_θ is defined by

$$T_t(u^n v^m) = e^{-t(n^2 + m^2)} u^n v^m \quad (n, m) \in \mathbb{Z}^2.$$

It is a τ -symmetric Markovian semigroup on the von Neumann algebra N_θ and the associated Dirichlet form on the standard Hilbert space $l^2(\mathbb{Z}^2)$ is given by

$$\mathcal{E}\left[\sum_{n,m \in \mathbb{Z}} \alpha_{n,m} u^n v^m\right] = \sum_{n,m \in \mathbb{Z}} (n^2 + m^2) |\alpha_{n,m}|^2$$

defined on the domain $\mathcal{F} \subset l^2(\mathbb{Z}^2)$ where the right hand side of the above formula is finite.

3.5. Dirichlet forms from derivations on C*-algebras. There exists a general interplay among Dirichlet forms and differential calculus on tracial C*-algebras (A, τ) (see [S 2,3], [CS1]) and obviously this provides a source of Dirichlet forms on von Neumann algebras (generated by A in the G.N.S. representation of the trace). Even if in this paper we use results of the theory pertaining to von Neumann algebras only, we notice that the nuance with the theory on C*-algebras relies, in the latter case, on the possibility to discuss *regularity*, a property on which further developments of potential theory are based ([CS4]).

Let (A, τ) be a separable C*-algebra endowed with a semifinite, faithful, lower semicontinuous trace. Denote by $\pi_\tau : A \rightarrow \mathcal{B}(L^2(A, \tau))$ the G.N.S. representation and consider the von Neumann algebra $N := (\pi_\tau(A))''$ acting on the G.N.S. space $L^2(A, \tau)$.

Let $(\mathcal{H}, \mathcal{J})$ be a symmetric Hilbert A -bimodule, i.e. a complex Hilbert space \mathcal{H} endowed with commuting continuous actions of A , on which and antiunitary operator \mathcal{J} intertwines the left and right actions

$$\mathcal{J}(a\xi b) = b^*(\mathcal{J}\xi)a^* \quad a, b \in A, \quad \xi \in \mathcal{H}.$$

A symmetric derivation $\partial : D(\partial) \rightarrow \mathcal{H}$ is a linear map defined on an involutive, dense *-subalgebra $D(\partial)$ of the Hilbert algebra $A \cap L^2(A, \tau)$, satisfying the Leibnitz rule

$$\partial(ab) = (\partial a)b + a(\partial b) \quad a, b \in D(\partial)$$

and the symmetry relation

$$\partial(a^*) = \mathcal{J}(\partial a) \quad a \in D(\partial).$$

Theorem 3.1. [CS1] *i) Assume the derivation $(\partial, D(\partial))$ to be a closable operator from $L^2(A, \tau)$ to \mathcal{H} . Then closure of the quadratic form*

$$\mathcal{E}[a] := \|\partial a\|_{\mathcal{H}}^2 \quad a \in \mathcal{F} := D(\partial)$$

is a Dirichlet form on $L^2(A, \tau)$ with respect to the von Neumann algebra with trace (N, τ) .

ii) Viceversa, any Dirichlet form on $L^2(A, \tau)$ whose domain is dense in A (regularity) arises in this way from an essentially unique derivation on A canonically associated with it.

Instances of the above structure can be found in all the situations illustrated so far. For example, in the case of the Dirichlet integral of a Riemannian manifold, the derivation is obviously given by the gradient operator. In case of Dirichlet forms on countable discrete groups associated to a c.n.d. function ℓ , the derivation is given by the multiplication by the 1-cocycle associated with ℓ (see [Boz]). Dirichlet forms associated to quantum stochastic Levy's processes on compact quantum groups, are represented by derivations given by Schürmann cocycles (see [CFK]). The structure of the derivation associated to the Dirac Laplacian on Riemannian manifolds with positive curvature operator plays a fundamental role in proving the results in [CS2]).

Dirichlet forms constructed by derivations emerge naturally in several other contexts.

3.6. Voiculescu's Dirichlet form in Free Probability. Here we describe succinctly a class of Dirichlet forms discovered in Free Probability Theory by D. V. Voiculescu (see [V1]). Let (M, τ) be a noncommutative probability space, i.e. a von Neumann algebra endowed with a faithful, normal, finite and normalized trace.

Let us fix a unital $*$ -subalgebra $1 \in B \subset M$ and a finite set $X := \{X_1, \dots, X_n\} \subset M$ of noncommutative random variables, i.e. self-adjoint elements of M , algebraically free with respect to B .

Let us consider the $*$ -subalgebra $B[X] \subset M$ generated by X and B (regarded as the algebra of noncommutative polynomials in the variables X with coefficients in the algebra B) and the von Neumann subalgebra $W \subset M$ generated by $B[X]$.

Let $HS(L^2(W, \tau)) \simeq \overline{L^2(W, \tau)} \otimes L^2(W, \tau)$ be the Hilbert W -bimodule of Hilbert-Schmidt operators on $L^2(W, \tau)$ and $1 \otimes 1 \in HS(L^2(W, \tau))$ the rank one projection onto the multiples of the unit $1 \in M \subset L^2(M, \tau)$.

Within this framework, D.V. Voiculescu introduced a natural differential calculus and an associated Dirichlet form which can be considered as an analogue in Free Probability of the Dirichlet integral of Euclidean domains.

Theorem 3.2. *There exist unique derivations $\partial_{X_i} : B[X] \rightarrow HS(L^2(W, \tau))$ such that*

$$\partial_{X_i} X_j = \delta_{ij} 1 \otimes 1, \quad \partial_{X_i} b = 0 \quad i = 1, \dots, n, \quad b \in B.$$

Under the assumption $1 \otimes 1 \in \text{dom}(\partial_{X_i}^)$ for all $i = 1, \dots, n$, it follows that*

- *$(\partial_{X_i}, B[X])$ is densely defined and closable in $L^2(W, \tau)$ for all $i = 1, \dots, n$,*
- *the closure of the densely defined quadratic form*

$$\mathcal{E}_X[a] := \sum_{i=1}^n \|\partial_{X_i} a\|_{\text{HS}}^2 \quad a \in \mathcal{F}_X := B[X]$$

is a Dirichlet form on $L^2(W, \tau)$.

Relevant aspects connected to the Dirichlet form above are the *Noncommutative Hilbert Transform* of X with respect to B , the *Free Fischer information* and the *Free Entropy*.

3.7. Voiculescu's Dirichlet form in BDF-theory. In his work [V2] about an extension of the Brown-Douglas-Fillmore theory for almost normal operators, mod the Hilbert-Schmidt ones, D.V. Voiculescu discussed the role played by a specific Dirichlet space $\mathcal{K}\Lambda(\Omega)$ on the C^* -algebra of compact operators on the Lebesgue space $L^2(\Omega, \lambda)$ over a measurable subset $\Omega \subset \mathbb{C}$ of the plane. Both the derivation and the Markovian semigroup are explicitly computed on a form core.

3.8. Transverse differential calculus of Riemannian foliation. In the construction of the transverse heat semigroup of a Riemannian foliation in [S4], a fundamental role is played by the *transverse differentiation*, a derivation on the C^* -algebra of the holonomy groupoid of the foliation.

4. AMENABILITY OF σ -FINITE VON NEUMANN ALGEBRAS

In this section we relate a certain characteristic of the spectrum of a Dirichlet form to the amenability of the von Neumann algebra. Recall that a von Neumann algebra N is said to

be *amenable* if, for every normal dual Banach N -bimodule X , the derivations $\delta : N \rightarrow X$ are all inner, i.e. they have the form

$$\delta(x) = x\xi - \xi x \quad x \in N$$

from some vector $\xi \in X$. It is a remarkable fact, and the byproduct of a tour de force that this property is equivalent to several others of apparently completely different nature, such as *hyperfiniteness*, *injectivity*, *semi-discreteness*, *Schwartz's property P*, *Tomiyama property E*. We refer to [Co2 Ch. V] for a review on these connections. Among the main examples of amenable von Neumann algebras, we recall: the von Neumann algebra of a locally compact amenable group, the crossed product of an abelian von Neumann algebra by an amenable locally compact group, the commutant von Neumann algebra of any continuous unitary representation of a connected locally compact group, the von Neumann algebra generated by any representation of a nuclear C^* -algebra.

4.1. Standard form of the spatial tensor product of von Neumann algebras. Here we summarize the well known construction of the standard form of the spatial tensor product of two von Neumann algebras in terms of Hilbert-Schmidt operators, mainly with the intention to make precise, in the next section, some properties of the symmetric embedding of a product state. For the reader's convenience we provide the simple proofs (see [T]).

Lemma 4.1. *Let $N \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A vector $\xi \in H$ is cyclic for the commutant N' if and only if it is separating for N .*

Proof. Suppose that for $x \in N$ we have $x\xi = 0$. Then $0 = x'x\xi = xx'\xi$ for all $x' \in N'$ so that $x\eta = 0$ for all $\eta \in \mathcal{M}'\xi$. As by assumption $\mathcal{M}'\xi$ is dense in H we get $x = 0$. \square

Lemma 4.2. *Let $N_k \subseteq \mathcal{B}(H_k)$ $k = 1, 2$ be von Neumann algebras. If the vectors $\xi_k \in H_k$, $k = 1, 2$ are cyclic for N_k , then the vector $\xi_1 \otimes \xi_2 \in H_1 \otimes H_2$ is cyclic for the spatial tensor product $N_1 \overline{\otimes} N_2$.*

Proof. By assumption $N_k\xi_k$ is dense in H_k for $k = 1, 2$. Hence $(N_1 \odot N_2)\xi_1 \otimes \xi_2$ is dense in $H_1 \otimes H_2$. As $(N_1 \overline{\otimes} N_2)\xi_1 \otimes \xi_2 \supseteq (N_1 \odot N_2)\xi_1 \otimes \xi_2$ we have that $\xi_1 \otimes \xi_2$ is cyclic for $N_1 \overline{\otimes} N_2$. \square

Lemma 4.3. *Let N_k $k = 1, 2$ be von Neumann algebras and $L^2(N_k)$ their standard forms. If the vectors $\xi_k \in L^2_+(N_k)$ $k = 1, 2$ are cyclic for N_k (hence separating) then the vector $\xi_1 \otimes \xi_2 \in L^2(N_1) \otimes L^2(N_2)$ is cyclic and separating for the spatial tensor product $N_1 \overline{\otimes} N_2$.*

Proof. The cyclicity of $\xi_1 \otimes \xi_2$ follows from Lemma 4.2. To prove separability, it is enough, by Lemma 4.1, to show that $\xi_1 \otimes \xi_2$ is cyclic for the commutant $(N_1 \overline{\otimes} N_2)'$. By Tomita's commutant Theorem (see [T]) $(N_1 \overline{\otimes} N_2)' = N_1' \overline{\otimes} N_2'$. Again by Lemma 4.2, it is enough to justify that ξ_k is cyclic for N_k' $k = 1, 2$ and this follows from the positivity of the vectors ξ_1 and ξ_2 . \square

4.2. Symmetric embedding of von Neumann algebras. Let N be a σ -finite von Neumann algebra and $\omega \in N_{*,+}$ a faithful, normal state.

In the standard form $(N, L^2(N, \omega), L^2_+(N, \omega))$, we denote by $\xi_\omega \in L^2_+(N, \omega)$ the cyclic vector representing the state ω and by J_ω and Δ_ω the associated modular conjugation and modular operator, respectively.

The symmetric embedding $i_\omega : N \rightarrow L^2(N, \omega)$ is defined by $i_\omega(x) := \Delta_\omega^{\frac{1}{4}} x \xi_\omega$ for $x \in N$. It is a completely positive contraction with dense range, which is also continuous between the weak*-topology of N and the weak topology of $L^2(N, \omega)$ and which is an order isomorphism of completely ordered sets between $\{x = x^* \in N : 0 \leq x \leq 1_N\}$ and $\{\xi = J_\omega \xi \in L^2(N, \omega) : 0 \leq \xi \leq \xi_\omega\}$ (see [Ara], [Co1], [Haa1] and [BR]).

Lemma 4.4. *Let N_k $k = 1, 2$ be von Neumann algebras and $L^2(N_k)$ their standard forms. Consider the cyclic (hence separating) vectors $\xi_k \in L^2(N_k)$ $k = 1, 2$ and the cyclic and separating vector $\xi_1 \otimes \xi_2 \in H_1 \otimes H_2$ for the spatial tensor product $N_1 \overline{\otimes} N_2$. Let J_k, Δ_k be the modular conjugation and the modular operator associated to $\xi_k \in H_k$ $k = 1, 2$ and $J_{\xi_1 \otimes \xi_2}, \Delta_{\xi_1 \otimes \xi_2}$ be the modular conjugation and the modular operator associated to $\xi_1 \otimes \xi_2$. Then the following identifications holds true*

- $J_{\xi_1 \otimes \xi_2} = J_{\xi_1} \otimes J_{\xi_2}$;
- $N_1 \xi_1 \odot N_2 \xi_2 \subseteq H_1 \otimes H_2$ is a core for the closed operator $\Delta_{\xi_1 \otimes \xi_2}^{\frac{1}{2}}$;
- $\Delta_{\xi_1 \otimes \xi_2}^{\frac{1}{2}}(\eta_1 \otimes \eta_2) = \Delta_{\xi_1}^{\frac{1}{2}}(\eta_1) \otimes \Delta_{\xi_2}^{\frac{1}{2}}(\eta_2)$ for $\eta_k \in N_k \xi_k$ and $k = 1, 2$.

We will denote by N° the opposite algebra of N : it coincides with N as a vector space but the product is taken in the reverse order $x^\circ y^\circ := (yx)^\circ$ for $x^\circ, y^\circ \in N^\circ$. As customary, we adopt the convention that elements $y \in N$, when regarded as elements of the opposite algebra are denoted by $y^\circ \in N^\circ$.

A linear functional ω on N , when considered as a linear functional on the opposite algebra N° is denoted by ω° and called the *opposite* of ω . As N and N° share the same positive cone, if ω is positive on N so is ω° on N° and if ω is normal so does its opposite.

By the properties of standard forms of von Neumann algebras, it follows that for the standard form $(N^\circ, L^2(N^\circ, \omega^\circ), L_+^2(N^\circ, \omega^\circ))$ of N° one has the following identifications

$$L^2(N^\circ, \omega^\circ) = L^2(N, \omega), \quad L_+^2(N^\circ, \omega^\circ) = L_+^2(N, \omega), \quad J_\omega = J_{\omega^\circ}, \quad \Delta_{\omega^\circ} = \Delta_\omega^{-1}, \quad \xi_{\omega^\circ} = \xi_\omega.$$

Using the isomorphism between N° and the commutant N' , given by $N^\circ \ni y^\circ \rightarrow J_\omega y^* J_\omega \in N'$, we can regard $L^2(N, \omega)$ not only as a left N -module but also as a left N° -module, hence as a right N -module and finally as a N - N -bimodule

$$y^\circ \xi := J_\omega y^* J_\omega \xi, \quad \xi y := J_\omega y^* J_\omega \xi, \quad x \xi y := x J_\omega y^* J_\omega \xi \quad x, y \in N, \xi \in L^2(N, \omega).$$

The symmetric embeddings associated to ω and ω° are related by

$$\begin{aligned} i_{\omega^\circ}(y^\circ) &= \Delta_{\omega^\circ}^{\frac{1}{4}}(\xi_\omega y) = \Delta_{\omega^\circ}^{\frac{1}{4}} J_\omega y^* J_\omega \xi_\omega = \Delta_\omega^{-\frac{1}{4}} \Delta_\omega^{\frac{1}{2}} y \xi_\omega = \Delta_\omega^{\frac{1}{4}} y \xi_\omega = i_\omega(y). \\ J_\omega(i_\omega(y^*)) &= J_\omega \Delta_\omega^{\frac{1}{4}}(y^* \xi_\omega) = J_\omega \Delta_\omega^{\frac{1}{4}} J_\omega \Delta_\omega^{\frac{1}{2}}(y \xi_\omega) = \Delta_\omega^{\frac{1}{4}}(y \xi_\omega) = i_\omega(y) = i_{\omega^\circ}(y^\circ). \end{aligned}$$

4.3. Hilbert-Schmidt operators. An Hilbert-Schmidt operator T is a bounded operator on $L^2(N, \omega)$ such that $\text{Trace}_{L^2(N, \omega)}(T^* T) < +\infty$. It may be represented as

$$T\xi := \sum_{k=0}^{\infty} \mu_k(\eta_k | \xi) \xi_k \quad \xi \in L^2(N, \omega)$$

in terms of suitable orthonormal systems $\{\eta_k : k \in \mathbb{N}\}$, $\{\xi_k : k \in \mathbb{N}\} \subset L^2(N, \omega)$ and a sequence $\{\mu_k : k \in \mathbb{N}\} \subset \mathbb{C}$ such that $\sum_{k=0}^{\infty} |\mu_k|^2 < +\infty$. The set of Hilbert-Schmidt operators $HS(L^2(N, \omega))$ is an Hilbert space under the scalar product $(T_1 | T_2) := \text{Trace}_{L^2(N, \omega)}(T_1^* T_2)$.

Lemma 4.5. *The binormal representations $\pi_{\text{co}}^1, \pi_{\text{co}}^2, \pi_{\text{co}}^3$ of $N \otimes_{\text{max}} N^\circ$, characterized by*

$$\begin{aligned} \pi_{\text{co}}^1 : N \otimes_{\text{max}} N^\circ &\rightarrow \mathcal{B}(HS(L^2(N, \tau))) \\ \pi_{\text{co}}^1(x \otimes y^\circ)(T) &:= x T y \quad x, y \in N, \quad T \in HS(L^2(N, \tau)), \\ \pi_{\text{co}}^2 : N \otimes_{\text{max}} N^\circ &\rightarrow \mathcal{B}(L^2(N, \tau) \otimes \overline{L^2(N, \tau)}) \\ \pi_{\text{co}}^2(x \otimes y^\circ)(\xi \otimes \bar{\eta}) &:= x \xi \otimes \bar{\eta} y \quad x, y \in N, \quad \xi, \eta \in L^2(N, \tau) \end{aligned}$$

$$\begin{aligned}\pi_{\text{co}}^3 : N \otimes_{\max} N^\circ &\rightarrow \mathcal{B}(L^2(N, \tau) \otimes L^2(N, \tau)) \\ \pi_{\text{co}}^3(x \otimes y^\circ)(\xi \otimes \eta) &:= x\xi \otimes \eta y \quad x, y \in N, \quad \xi, \eta \in L^2(N, \tau),\end{aligned}$$

are unitarily equivalent by

$$\begin{aligned}U : L^2(N, \tau) \otimes \overline{L^2(N, \tau)} &\rightarrow L^2(N, \tau) \otimes L^2(N, \tau) & U(\xi \otimes \bar{\eta}) &:= \xi \otimes J_\omega \eta \\ V : L^2(N, \tau) \otimes \overline{L^2(N, \tau)} &\rightarrow \text{HS}(L^2(N, \tau)) & V(\xi \otimes \bar{\eta})(\zeta) &:= (\eta|\zeta)\xi.\end{aligned}$$

Collectively indicated by the symbol π_{co} , if no confusion can arise, it will be termed the coarse representation of the C^* -algebra $N \otimes_{\max} N^\circ$. It gives rise by weak closure

$$(\pi_{\text{co}}(N \otimes_{\max} N^\circ))'' = N \overline{\otimes} N^\circ$$

to the spatial tensor product of N by its opposite N° .

Lemma 4.6. *The normal extension of the coarse representation π_{co} of the C^* -algebra $N \otimes_{\max} N^\circ$ to the von Neumann tensor product $N \overline{\otimes} N^\circ$ is the standard representation of $N \overline{\otimes} N^\circ$ (and it will still denoted by the same symbol).*

The standard positive cones in the various equivalent representations are determined by

- $\text{HS}(L^2(N, \omega))_+$ is the set of all nonnegative Hilbert-Schmidt operators on $L^2(N, \omega)$
- $(L^2(N, \tau) \otimes \overline{L^2(N, \omega)})_+$ is generated by vectors $\xi \otimes \bar{\xi}$ with $\xi \in L^2(N, \omega)$
- $(L^2(N, \tau) \otimes L^2(N, \omega))_+$ is generated by vectors $\xi \otimes J_\omega \xi$ with $\xi \in L^2(N, \omega)$.

The standard Hilbert space and the positive cone of $N \overline{\otimes} N^\circ$ will be denoted also by

$$L^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ), \quad L_+^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ).$$

Lemma 4.7. *Let $T : L^2(N, \omega) \rightarrow L^2(N, \omega)$ be a bounded operator and consider on the involutive algebra $N \odot N^\circ$, the linear functional determined by*

$$\Theta_T : N \odot N^\circ \rightarrow \mathbb{C} \quad \Theta_T(x \otimes y^\circ) := (i_\omega(y^*)|T i_\omega(x)) \quad x \otimes y^\circ \in N \odot N^\circ.$$

Then Θ_T is a positive linear functional on $N \odot N^\circ$ if and only if T is completely positive.

Proof. i) The positive cone of $N \odot N^\circ$ is generated by elements of type $\nu^* \nu = \sum_{j,k=1}^n x_j^* x_k \otimes (y_j^* y_k)^\circ$ where $\nu = \sum_{k=1}^n x_k \otimes y_k^\circ \in N \odot N^\circ$. The result then follows by the identity

$$\Theta_T(\nu^* \nu) = \sum_{j,k=1}^n \Theta_T(x_j^* x_k \otimes (y_j^* y_k)^\circ) = \sum_{j,k=1}^n (i_\omega(y_k^* y_j)|T i_\omega(x_j^* x_k)),$$

the completely positivity of the symmetric embedding $i_\omega : N \rightarrow L^2(N, \tau)$ and the positivity of $[x_j^* x_k]_{j,k=1}^n$ in $\mathbb{M}_n(N)$. □

Lemma 4.8. *Let $T : L^2(N, \omega) \rightarrow L^2(N, \omega)$ be a completely positive operator and consider the positive linear functional Θ_T on $N \odot N^\circ$. Then, among the properties*

- a) Θ_T is a state on $N \odot N^\circ$
- b) T is a contraction
- c) $T \xi_\omega = \xi_\omega$

we have that the following relations

- i) a) and b) imply c) and $\|T\| = 1$
- ii) c) implies a) and b).

Proof. i) By a) and b) we have $1 = \Theta_T(1_N \otimes 1_{N^\circ}) = (\xi_\omega | T\xi_\omega) \leq \|\xi_\omega\| \cdot \|T\xi_\omega\| \leq \|\xi_\omega\|^2 \cdot \|T\| = 1$ that implies $\|T\| = \|T\xi_\omega\| = 1$ and $(\xi_\omega | T\xi_\omega) = \|\xi_\omega\| \cdot \|T\xi_\omega\|$ which provide $T\xi_\omega = \xi_\omega$. ii) The proof that c) implies a) is immediate while the proof that c) implies b) can be found in [C1]. \square

4.4. Spectral growth rate. In the following definition the notion of growth rate of a finitely generated, countable discrete group is extended to σ -finite von Neumann algebras having the Haagerup Property (H), i.e. von Neumann algebras admitting Dirichlet forms with discrete spectrum. The idea for this generalization results from the results of [CS5] (see discussion in Example 4.11 below).

Definition 4.9. (Spectral growth rate of Dirichlet forms). Let (N, ω) be a σ -finite, von Neumann algebra with a fixed faithful, normal state on it. To avoid trivialities we assume N to be *infinite dimensional*.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(N, \omega)$ and let $(L, D(L))$ be the associated nonnegative, self-adjoint operator. Assume that its spectrum $\sigma(L) = \{\lambda_k \geq 0 : k \in \mathbb{N}\}$ is *discrete*, i.e. its points are isolated eigenvalues of finite multiplicity (repeated in non decreasing order according to their multiplicities).

Then let us set

$$\Lambda_n := \{k \in \mathbb{N} : \lambda_k \in [0, n]\}, \quad \beta_n := \#(\Lambda_n), \quad n \in \mathbb{N}$$

and define the *spectral growth rate* of $(\mathcal{E}, \mathcal{F})$ as

$$\Omega(\mathcal{E}, \mathcal{F}) := \limsup_{n \in \mathbb{N}} \sqrt[n]{\beta_n}.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to have

- *exponential growth* if $\Omega(\mathcal{E}, \mathcal{F}) > 1$
- *subexponential growth* if $\Omega(\mathcal{E}, \mathcal{F}) = 1$
- *polynomial growth* if, for some $c, d > 0$, $\beta_n \leq c \cdot n^d$ for all $n \in \mathbb{N}$
- *intermediate growth* if it has subexponential growth but not polynomial growth.

Lemma 4.10. *Setting $\gamma_0 = \beta_0$ and*

$$\gamma_n := \beta_n - \beta_{n-1} = \#\{k \in \mathbb{N} : \lambda_k \in (n-1, n]\}, \quad n \in \mathbb{N},$$

and

$$\Omega'(\mathcal{E}, \mathcal{F}) := \limsup_{n \in \mathbb{N}^*} \sqrt[n]{\gamma_n}$$

we have

$$\Omega(\mathcal{E}, \mathcal{F}) = \Omega'(\mathcal{E}, \mathcal{F}) \geq 1.$$

Proof. On one hand, by definition, we have $\Omega(\mathcal{E}, \mathcal{F}) \geq \Omega'(\mathcal{E}, \mathcal{F})$. On the other hand, since, by assumption, N is infinite dimensional and $\sigma(L)$ is discrete, we have $\Omega(\mathcal{E}, \mathcal{F}) \geq \Omega'(\mathcal{E}, \mathcal{F}) \geq 1$. Consider now the following identity involving analytic functions in a neighborhood of $0 \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \beta_n z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \gamma_n z^n$$

and notice that the radius of convergence of the series on the left-hand side is $R = 1/\Omega(\mathcal{E}, \mathcal{F})$, while the radius of convergence of the series on the right-hand side is $R' = 1/\Omega'(\mathcal{E}, \mathcal{F})$ so that $R \leq R' \leq 1$. Since $(1-z)^{-1}$ is analytic in the open unit disk centered in $z = 0$, the above identity implies that $R \geq R'$ so that $\Omega(\mathcal{E}, \mathcal{F}) \leq \Omega'(\mathcal{E}, \mathcal{F})$. \square

Example 4.11. (Spectral growth rate on countable discrete groups).

i) On a countable discrete group Γ , if there exists a proper, c.n.d. function ℓ , then the associated Dirichlet form $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum $\sigma(L) = \{\ell(g) \in [0, +\infty) : g \in \Gamma\}$.

ii) On a finitely generated, countable discrete group Γ , if the length ℓ_S corresponding to a finite system of generators $S \subseteq \Gamma$ is negative definite, then the spectral growth rate $\Omega(\mathcal{E}_{\ell_S}, \mathcal{F}_{\ell_S})$ of the corresponding Dirichlet form coincides with growth rate of (Γ, S) (see [deH Ch. VI]).

iii) Moreover, if (Γ, S) has polynomial growth, it has been shown in [CS5] that there exists on Γ a proper, c.n.d. function ℓ with polynomial growth. The associated Dirichlet form $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ will have polynomial spectral growth rate.

Remark 4.12. By a well known bound (see [R Theorem 3.37])

$$1 \leq \liminf_n \frac{\beta_{n+1}}{\beta_n} \leq \limsup_{n \in \mathbb{N}} \sqrt[n]{\beta_n},$$

if the spectral growth rate is subexponential, then $\liminf_n \frac{\beta_{n+1}}{\beta_n} = 1$ so that there exists a subsequence of $\{\frac{\beta_{n+1}}{\beta_n}\}_{n \in \mathbb{N}}$ converging to 1. In other words, the sequence of spectral subspaces $\{E_n\}_{n \in \mathbb{N}}$ corresponding to the interval $[0, n] \subset [0, +\infty)$ admits a subsequence such that

$$\lim_k \frac{\dim E_{n_k+1}}{\dim E_{n_k}} = 1.$$

Subexponential growth can be equivalently stated in terms of the nuclearity of the completely Markovian semigroup $\{e^{-tL} : t > 0\}$ on $L^2(N, \omega)$:

Lemma 4.13. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ has subexponential spectral growth if and only if the Markovian semigroup $\{e^{-tL} : t > 0\}$ on $L^2(N, \omega)$ is nuclear or trace-class in the sense that:*

$$\text{Trace}(e^{-tL}) = \sum_{k \in \mathbb{N}} e^{-t\lambda_k} < +\infty \quad t > 0.$$

Proof. Since

$$\gamma_0 + \sum_{n \in \mathbb{N}^*} \gamma_n e^{-tn} \leq \sum_{k \in \mathbb{N}} e^{-t\lambda_k} \leq \gamma_0 + e^t \sum_{n \in \mathbb{N}^*} \gamma_n e^{-tn} \quad t > 0,$$

the series $\sum_{k \in \mathbb{N}} e^{-t\lambda_k}$ and $\sum_{n \in \mathbb{N}^*} \gamma_n e^{-tn}$ converge or diverge simultaneously. They obviously converge for all $t > 0$ if and only if $\Omega'(\mathcal{E}, \mathcal{F}) \leq 1$. \square

Example 4.14. If on a countable discrete group Γ , there exists a c.n.d. function ℓ , such that $\sum_{g \in \Gamma} e^{-t\ell(g)} < +\infty$ for all $t > 0$, then ℓ is proper, the spectrum of the associated Dirichlet form $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ coincides with $\{\ell(g) \in [0, +\infty) : g \in \Gamma\}$ and it is thus discrete with subexponential growth.

The following is the main result of this section.

Theorem 4.15. *Let (N, ω) be a σ -finite von Neumann algebra endowed with a normal, faithful state on it. If there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \omega)$ having subexponential spectral growth, then N is amenable.*

Proof. Recall that N is amenable if and only if the identity or standard bimodule ${}_N L^2(N)_N$ is weakly contained in the coarse or Hilbert-Schmidt bimodule \mathcal{H}_{co} (see [Po1]). Consider the completely positive semigroup $\{T_t := e^{-tL} : t > 0\}$ and assume, for simplicity, that the

cyclic vector is invariant: $T_t \xi_\omega = \xi_\omega$ for all $t > 0$. Recall (cf. Lemma 4.7) that the complete positivity of T_t provides a binormal state on $N \otimes_{\max} N^\circ$ characterized by

$$\Phi_t : N \otimes_{\max} N^\circ \rightarrow \mathbb{C} \quad \Phi_t(x \otimes y^\circ) := (i_\omega(y^*) | T_t i_\omega(x)).$$

To compute this state, we consider the spectral representation $T_t = \sum_{k \geq 0} e^{-t\lambda_k} P_k$ (converging strongly) in terms of the rank-one projections P_k on $L^2(N, \omega)$ associated to each eigenvalue λ_k (repeated according to their multiplicity). Notice that by Markovianity, the semigroup commutes with the modular conjugation J_ω so that each eigenvector ξ_k may be assumed to be real: $\xi_k = J_\omega \xi_k$. We then have

$$\begin{aligned} \Phi_t(x \otimes y^\circ) &= (i_\omega(y^*) | T_t i_\omega(x)) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (i_\omega(y^*) | P_k(i_\omega(x))) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (i_\omega(y^*) | (\xi_k | i_\omega(x)) \xi_k) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (\xi_k | i_\omega(x)) (i_\omega(y^*) | \xi_k). \end{aligned}$$

As the series $Z_t := \sum_{k=0}^{\infty} e^{-t\lambda_k} \xi_k \otimes \xi_k$ is norm convergent for all $t > 0$ by the nuclearity of the semigroup, since J_ω is an antiunitary operator on $L^2(N)$, using Lemma 4.4 above we have

$$\begin{aligned} \Phi_t(x \otimes y^\circ) &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (\xi_k | i_\omega(x)) (J_\omega \xi_k | J_\omega i_\omega(y^*)) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (\xi_k | i_\omega(x)) (\xi_k | i_{\omega^\circ}(y^\circ)) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} (\xi_k \otimes \xi_k | i_\omega(x) \otimes i_{\omega^\circ}(y^\circ))_{L^2(N, \omega) \otimes L^2(N, \omega)} \\ &= \left(\sum_{k=0}^{\infty} e^{-t\lambda_k} \xi_k \otimes \xi_k \middle| i_\omega(x) \otimes i_{\omega^\circ}(y^\circ) \right)_{L^2(N, \omega) \otimes L^2(N, \omega)} \\ &= \left(Z_t \middle| i_{\omega \otimes \omega^\circ}(x \otimes y^\circ) \right)_{L^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ)}. \end{aligned}$$

Since, by construction, Φ_t is a normal functional and since the symmetric embeddings of von Neumann algebras are continuous when N is endowed with the weak*-topology and $L^2(N)$ is endowed with the weak topology, by continuity we have

$$\Phi_t(\zeta) = \left(Z_t \middle| i_{\omega \otimes \omega^\circ}(\zeta) \right)_{L^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ)} \quad \zeta \in N \overline{\otimes} N^\circ.$$

In other words, the linear functional Φ_t extends as a σ -weakly continuous linear functional on the spatial tensor product $N \overline{\otimes} N^\circ$. Φ_t being positive by construction, there exist a unique positive element $\Omega_t \in L_+^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ)$ such that

$$\Phi_t(z) = (i_\omega(y^*) | T_t i_\omega(x)) = \left(\Omega_t | \pi_{\text{co}}(z) \Omega_t \right)_{L^2(N \overline{\otimes} N^\circ, \omega \otimes \omega^\circ)} \quad z \in N \overline{\otimes} N^\circ.$$

and the GNS representation of $N \otimes_{\max} N^\circ$ associated to Φ_t coincides with a sub-representation of π_{co} . In other words, the $N - N$ -correspondence \mathcal{H}_t associated to the completely positive

map T_t is contained in the coarse $N-N$ -correspondence \mathcal{H}_{co} for all $t > 0$. Since the semigroup $\{T_t : t > 0\}$ is strongly continuous on $L^2(N, \omega)$, for all $x \otimes y^\circ \in N \otimes_{\max} N^\circ$ we have

$$\begin{aligned}
\lim_{t \downarrow 0} \left(\Omega_t | \pi_{\text{co}}(x \otimes y^\circ) \Omega_t \right)_{L^2(N, \omega) \otimes L^2(N, \omega)} &= (i_\omega(y^*) | i_\omega(x))_{L^2(N, \omega)} \\
&= (\Delta_\omega^{\frac{1}{4}} y^* \xi_\omega | \Delta_\omega^{\frac{1}{4}} x \xi_\omega) \\
&= (\Delta_\omega^{\frac{1}{2}} y^* \xi_\omega | x \xi_\omega) \\
&= (J_\omega y \xi_\omega | x \xi_\omega) \\
&= (J_\omega y J_\omega \xi_\omega | x \xi_\omega) \\
&= (J_\omega y J_\omega \xi_\omega | J_\omega y^* J_\omega x \xi_\omega) \\
&= (J_\omega y J_\omega \xi_\omega | x J_\omega y^* J_\omega \xi_\omega) \\
&= (\xi_\omega | x \xi_\omega y) \\
&= (\xi_\omega | \pi_{\text{id}}(x \otimes y^\circ) \xi_\omega)
\end{aligned}$$

and by continuity

$$\lim_{t \downarrow 0} \left(\Omega_t | \pi_{\text{co}}(z) \Omega_t \right)_{L^2(N, \omega) \otimes L^2(N, \omega)} = (\xi_\omega | \pi_{\text{id}}(z) \xi_\omega) \quad z \in N \otimes_{\max} N^\circ.$$

This proves that the identity correspondence \mathcal{H}_{id} is weakly contained in the coarse correspondence \mathcal{H}_{co} and thus N is amenable at least if the semigroup leaves the cyclic vector invariant.

To deal with the general case, remark first that, by strong continuity, we have that $\lim_{t \downarrow 0} (\xi_\omega | T_t \xi_\omega) = \|\xi_\omega\|^2 = 1$ and there exist $t_0 > 0$ such that $(\xi_\omega | T_t \xi_\omega) > 0$ for all $0 < t < t_0$. Applying the argument above to the binormal states

$$\Phi_t(x \otimes y^\circ) := \frac{1}{(\xi_\omega | T_t \xi_\omega)} (i_\omega(y^*) | T_t i_\omega(x)) \quad x \otimes y^\circ \in N \otimes_{\max} N^\circ, \quad 0 < t < t_0$$

we get the amenability of N even in the general situation. \square

Corollary 4.16. *If the von Neumann algebra N is not amenable, then any Dirichlet form $(\mathcal{E}, \mathcal{F})$ with respect to any normal, faithful state ω has exponential growth rate $\Omega(\mathcal{E}, \mathcal{F}) > 1$, i.e. its sequence of eigenvalues has exponentially growing distribution.*

The following one is a generalization of a result of Guentner-Kaminker [GK].

Corollary 4.17. *Let Γ be a countable discrete group, $\lambda : \Gamma \rightarrow \mathcal{B}(l^2(\Gamma))$ be its left regular representation, $L(\Gamma)$ its associated von Neumann algebra and τ its trace state. If there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(L(\Gamma), \tau)$ having subexponential spectral growth, then the group Γ is amenable.*

Proof. Under the assumptions, the group von Neumann algebra $L(\Gamma)$ is amenable by the above theorem. Hence by a well known result of A. Connes, the group Γ is amenable. \square

Example 4.18. (Free orthogonal quantum groups) On the von Neumann algebra $L^\infty(O_2^+, \tau)$ of the free orthogonal quantum group O_2^+ with respect to its Haar state τ , it has been constructed in [CFK] a Dirichlet form with an explicitly computed discrete spectrum of polynomial growth (and spectral dimension $d := \limsup_n \frac{\ln \beta_n}{\ln n} = 3$). Applying the theorem above we get an independent proof of the amenability of $L^\infty(O_2^+, \tau)$, a result which has been proved by M. Brannan [Bra].

5. RELATIVE AMENABILITY OF INCLUSIONS OF FINITE VON NEUMANN ALGEBRAS

In this section we extend the previous result to the *relative amenability* of inclusions of finite von Neumann algebras $B \subseteq N$, as defined by S. Popa [Po 1,2]. This extension is based on the properties of the relative tensor product of Hilbert bimodules and on the properties of the *basic construction*, which we will presently recall (see [Chr], [GHJ], [J], [SiSm]).

5.1. Basic construction of finite inclusions. Let N be a von Neumann algebra admitting a normal faithful trace state τ and $1_N \in B \subseteq N$ a von Neumann subalgebra with the same identity (see [Chr], [J1], [Po1], [PiPo], [SiSm]).

Recall that the relative tensor product $L^2(N, \tau) \otimes_B L^2(N, \tau)$ over B of the N - B -bimodule ${}_N L^2(N, \tau)_B$ by the B - N -bimodule ${}_B L^2(N, \tau)_N$, constructed in [S1], is isomorphic, as an N - N -bimodule, to the N - N -correspondence \mathcal{H}_B associated to the conditional expectation $E_B : N \rightarrow B$ from N onto B . The latter being generated by the GNS construction applied to the binormal state

$$\Phi_B : N \otimes_{\max} N^\circ \rightarrow \mathbb{C} \quad \Phi_B(x \otimes y^\circ) := \tau(E_B(x)y).$$

Let e_B be the projection in $\mathcal{B}(L^2(N, \tau))$ onto $L^2(B, \tau)$ and consider the *basic construction* $\langle N, B \rangle$, i.e. the von Neumann algebra in $\mathcal{B}(L^2(N, \tau))$ generated by N and the projection e_B . For example, if $B = \mathbb{C}1_N$ then $\langle N, B \rangle = \mathcal{B}(L^2(N, \tau))$ and when $B = N$ then $\langle N, B \rangle = N$. Denoting by $\xi_\tau \in L^2(N, \tau)$ the cyclic vector representing τ one has

$$e_B(x\xi_\tau) = E_B(x)\xi_\tau, \quad e_B x e_B = E_B(x)e_B \quad x \in N.$$

It can be shown that an element $x \in N$ commutes with the projection e_B if and only if $x \in B$. Moreover, $\text{span}(Ne_B N)$ is weakly*-dense in $\langle N, B \rangle$ and $e_B \langle N, B \rangle e_B = Be_B$. It can be shown that

$$\langle N, B \rangle = (JBJ)' \subseteq \mathcal{B}(L^2(N, \tau))$$

so that $\langle N, B \rangle$ is semifinite since B is finite. In particular, there exists a unique normal, semifinite faithful trace Tr characterized by

$$\text{Tr}(xe_B y) = \tau(xy) \quad x, y \in N.$$

and there exists also a unique N - N -bimodule map Φ from $\text{span}(Ne_B N)$ into N satisfying

$$\Phi(xe_B y) = xy \quad x, y \in N$$

and $\text{Tr} = \tau \circ \Phi$. The map Φ extends to a contraction between the N - N -bimodules $L^1(\langle N, B \rangle, \text{Tr})$ and $L^1(N, \tau)$ and satisfies

$$e_B X = e_B \Phi(e_B X) \quad X \in \langle N, B \rangle.$$

Moreover, $\Phi(e_B X) \in L^2(\langle N, B \rangle, \text{Tr})$ for all $X \in \langle N, B \rangle$. These properties enable us to prove that the identity correspondence $L^2(\langle N, B \rangle, \text{Tr})$ of the algebra $\langle N, B \rangle$ reduces to the relative correspondence \mathcal{H}_B when restricted to the subalgebra $N \subseteq \langle N, B \rangle$.

Proposition 5.1. *The N - N -correspondences \mathcal{H}_B and $L^2(\langle N, B \rangle, \text{Tr})$ are isomorphic. In particular, the binormal state is given by*

$$\Phi_B(x \otimes y^\circ) = (e_B | x e_B y)_{L^2(\langle N, B \rangle, \text{Tr})} \quad x, y \in N$$

so that the cyclic vector representing the state Φ_B is $e_B \in L^2(\langle N, B \rangle, \text{Tr})$.

Proof. Let us consider the map defined on the domain

$$D(\Psi) := \text{span}\{[x \otimes y^\circ]_{\mathcal{H}_B} : x, y \in N\}$$

by

$$\Psi : D(\Psi) \rightarrow L^2(\langle N, B \rangle, \text{Tr}) \quad \Psi([x \otimes y^\circ]_{\mathcal{H}_B}) := xe_By \quad x, y \in N.$$

Here $[x \otimes y^\circ]_{\mathcal{H}_B}$ denotes the element of \mathcal{H}_B image of the elementary tensor product $x \otimes y^\circ$, in the GNS construction of the state Φ_B . The map is well defined because $\|e_B\|^2 = \text{Tr}(e_B) = \tau(\Phi(e_B)) = \tau(1_N) = 1$ and $\|xe_By\|_2 \leq \|x\| \cdot \|y\| \cdot \|e_B\|_2 = \|x\| \cdot \|y\|$. By the definition of the Hilbert space \mathcal{H}_B , the map Ψ is densely defined. For $x, y \in N$ we have

$$\begin{aligned} \|\Psi([x \otimes y^\circ]_{\mathcal{H}_B})\|_2^2 &= \|xe_By\|_2^2 \\ &= \text{Tr}((xe_By)^*(xe_By)) \\ &= \text{Tr}(y^*e_Bx^*xe_By) \\ &= \text{Tr}(y^*e_Be_Bx^*xe_Be_By) \\ &= \text{Tr}((e_Bx^*xe_B)(e_Byy^*e_B)) \\ &= \text{Tr}(E_B(x^*x)e_BE_B(yy^*)e_B) \\ &= \text{Tr}(E_B(x^*x)e_BE_B(yy^*)) \\ &= \tau(\Phi(E_B(x^*x)e_BE_B(yy^*))) \\ &= \tau(E_B(x^*x)E_B(yy^*)) \\ &= \tau(E_B(E_B(x^*x)yy^*)) \\ &= \tau(E_B(x^*x)yy^*) \\ &= \Phi_B(x^*x \otimes (yy^*)^\circ) \\ &= \Phi_B(x^*x \otimes (y^*)^\circ y^\circ) \\ &= \Phi_B((x^* \otimes (y^*)^\circ)(x \otimes y^\circ)) \\ &= \Phi_B((x \otimes y^\circ)^*(x \otimes y^\circ)) \\ &= \|[x \otimes y^\circ]_{\mathcal{H}_B}\|_{\mathcal{H}_B}^2. \end{aligned}$$

By polarization, for all $\{x_j\}_{j=1}^n \subset N$ we have also

$$(\Psi([x_j \otimes y_j^\circ]_{\mathcal{H}_B}) | \Psi([x_k \otimes y_k^\circ]_{\mathcal{H}_B}))_2 = ([x_j \otimes y_j^\circ]_{\mathcal{H}_B} | [x_k \otimes y_k^\circ]_{\mathcal{H}_B})_{\mathcal{H}_B} \quad j, k = 1, \dots, n.$$

Consider now $\nu = \sum_{k=1}^n [x_k \otimes y_k^\circ]_{\mathcal{H}_B} \in D(\Psi)$ so that $\nu^*\nu = \sum_{j,k=1}^n [x_j^*x_k \otimes (y_ky_j^*)^\circ]_{\mathcal{H}_B} \in D(\Psi)$ and then

$$\|\Psi(\nu)\|_2^2 = \sum_{j,k=1}^n ([x_j \otimes y_j^\circ]_{\mathcal{H}_B} | [x_k \otimes y_k^\circ]_{\mathcal{H}_B})_{\mathcal{H}_B} = (\nu | \nu)_{\mathcal{H}_B} = \|\nu\|_{\mathcal{H}_B}^2.$$

Hence the map Ψ extends to an isometry from \mathcal{H}_B into $L^2(\langle N, B \rangle, \text{Tr})$ which is clearly an $N - N$ -bimodule map. Since $\text{Im}(\Psi) = \text{span}(Ne_BN)$ is weakly*-dense in $\langle N, B \rangle$, it is also dense in $L^2(\langle N, B \rangle, \text{Tr})$. By the isometric property we have that $\text{Im}(\Psi)$ is closed so that Ψ is a surjective isometry. Finally, for $x, y \in N$ we compute

$$\begin{aligned} (e_B | xe_By)_{L^2(\text{Tr})} &= \text{Tr}(e_Bxe_By) \\ &= \text{Tr}(E_B(x)e_By) \\ &= \tau(\Phi(E_B(x)e_By)) \\ &= \tau(E_B(x)y) \\ &= \Phi_B(x \otimes y^\circ). \end{aligned}$$

□

Definition 5.2. (B -invariant Dirichlet forms). Let N be a von Neumann algebra admitting a normal faithful tracial state τ and $1_N \in B \subseteq N$ a von Neumann subalgebra. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ is said to be a B -invariant if

$$b\mathcal{F} \subseteq \mathcal{F}, \quad \mathcal{E}(b\xi|\xi) = \mathcal{E}(\xi|b^*\xi) \quad b \in B, \quad \xi \in \mathcal{F}$$

and

$$\mathcal{F}b \subseteq \mathcal{F}, \quad \mathcal{E}(\xi b|\xi) = \mathcal{E}(\xi|b^*\xi) \quad b \in B, \quad \xi \in \mathcal{F}.$$

Since, by definition, a Dirichlet form is J -real, the above two properties are in fact equivalent.

In terms of the associated nonnegative, self-adjoint operator $(L, D(L))$, B -invariance means that the resolvent family $\{(\lambda + L)^{-1} : \lambda > 0\}$ is B -bimodular for some and hence all $\lambda > 0$

$$\begin{aligned} (\lambda + L)^{-1}(b\xi) &= b((\lambda + L)^{-1}\xi) \\ (\lambda + L)^{-1}(\xi b) &= ((\lambda + L)^{-1}\xi)b \quad \xi \in L^2(N, \tau), b \in B, \end{aligned}$$

or that, alternatively, the semigroup $\{e^{-tL} : t > 0\}$ is for some and hence all $t > 0$ a B -bimodular map

$$\begin{aligned} e^{-tL}(b\xi) &= b(e^{-tL}\xi) \\ e^{-tL}(\xi b) &= (e^{-tL}\xi)b \quad \xi \in L^2(N, \tau), b \in B. \end{aligned}$$

Since the Markovianity of the Dirichlet form implies that the semigroup and the resolvent commute with the modular conjugation J , we have that the B -invariance of the Dirichlet form provides that the semigroup and the resolvent belong to the relative commutant of B in the basic construction $\langle N, B \rangle$:

$$(\lambda + L)^{-1}, e^{-tL} \in (JBJ)' \cap B' = \langle N, B \rangle \cap B' \quad t > 0, \quad \lambda > 0.$$

Definition 5.3. (Relative discrete spectrum) We say that $(\mathcal{E}, \mathcal{F})$ or $(L, D(L))$ have *discrete spectrum relative to the inclusion $B \subseteq N$* if the Markovian semigroup, or equivalently the resolvent, belongs to the *compact ideal space* $\mathcal{J}(\langle N, B \rangle)$ ([J], [Po2], [SiSm]) of the basic construction, generated by projections in $\langle N, B \rangle$ having finite trace:

$$\begin{aligned} e^{-tL} &\in \mathcal{J}(\langle N, B \rangle) \quad \text{for some and hence all } t > 0, \\ (\lambda + L)^{-1} &\in \mathcal{J}(\langle N, B \rangle) \quad \text{for some and hence all } \lambda > 0, \end{aligned}$$

Definition 5.4. (Relative spectral growth rate of Dirichlet forms) Let N be a von Neumann algebra admitting a normal faithful tracial state τ and $1_N \in B \subseteq N$ a von Neumann subalgebra. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ which is B -invariant is said to be have

- *exponential spectral growth relative to $B \subseteq N$* if $\text{Tr}(e^{-tL}) = +\infty$ for some $t > 0$;
- *subexponential spectral growth relative to $B \subseteq N$* if $\text{Tr}(e^{-tL}) < +\infty$ for all $t > 0$.

Remark 5.5. Let E^L be the spectral measure of the self-adjoint operator $(L, D(L))$. If the Dirichlet form is B -invariant then E^L takes its values in the class of projections of the von Neumann algebra $\langle N, B \rangle$ and we can consider the real measure $\nu_B := \text{Tr} \circ E^L$ on $[0, +\infty)$, supported by the spectrum $\sigma(L)$.

The subexponential spectral growth condition relative to $B \subseteq N$ then means that the Laplace Transform $\hat{\nu}_B$ of the measure ν_B has abscissa of convergence equal to $\lambda(\nu_B) = 0$. Whenever the measure ν_B has the meaning of a *density of states relatively to B* , the measure $\mu_B(d\lambda) := \lambda \nu_B(d\lambda)$ acquires the meaning of *spectral energy density relatively to B* and the following

identity holds true $\frac{d\hat{\nu}_B}{dt}(t) = -\hat{\mu}_B(t)$ for $t > 0$. In Quantum Statistical Mechanics $\hat{\nu}_B$ is called the *partition function* and $\Phi_\beta(A) := \frac{\text{Tr}(Ae^{-\beta L})}{\text{Tr}(e^{-\beta L})}$ the Gibbs state at inverse temperature $\beta > 0$.

The following is the main result of this section.

Theorem 5.6. *Let N be a von Neumann algebra admitting a normal faithful tracial state τ and $1_N \in B \subseteq N$ a von Neumann subalgebra.*

If there exists a B -invariant Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ having subexponential spectral growth relatively to $B \subseteq N$, then the inclusion $B \subseteq N$ is amenable.

Proof. Let us check first the following identity

$$(T^*|xe_B y)_{L^2(\text{Tr})} = (i_\tau(y^*)|T(i_\tau(x)))_{L^2(\tau)} \quad T \in \langle N, B \rangle \cap L^2(\langle N, B \rangle, \text{Tr}), \quad x, y \in N.$$

As $\text{span}(Ne_B N)$ is weakly* dense in $\langle N, B \rangle$, it is enough to prove the identity for $T \in Ne_B N$. If $T = ue_B v$ for some $u, v \in N$ we have

$$e_B y T x e_B = e_B y u e_B v x e_B = (e_B y u e_B)(e_B v x e_B) = E_B(yu) e_B E_B(vx) e_B$$

and then

$$\begin{aligned} (T^*|xe_B y)_{L^2(\text{Tr})} &= \text{Tr}(T x e_B y) \\ &= \text{Tr}(e_B y T x e_B) \\ &= \tau(\Phi(e_B y T x e_B)) \\ &= \tau(\Phi(E_B(yu) e_B E_B(vx) e_B)) \\ &= \tau(E_B(yu) \Phi(e_B E_B(vx) e_B)) \\ &= \tau(E_B(yu) \Phi(E_B(vx) e_B)) \\ &= \tau(E_B(yu) E_B(vx)) \\ &= \tau(E_B(yu E_B(vx))) \\ &= \tau(yu E_B(vx)) \\ &= (u^* y^* \xi_\tau | E_B(vx) \xi_\tau)_{L^2(\tau)} \\ &= (i_\tau(y^*) | u E_B(vx) \xi_\tau)_{L^2(\tau)} \\ &= (i_\tau(y^*) | u e_B(v(x \xi_\tau)))_{L^2(\tau)} \\ &= (i_\tau(y^*) | u e_B v(i_\tau(x)))_{L^2(\tau)} \\ &= (i_\tau(y^*) | T(i_\tau(x)))_{L^2(\tau)} \end{aligned}$$

so that the identity holds true. Under the hypothesis of subexponential spectral growth, we have that $T_t := e^{-tL} \in L^2(\langle N, b \rangle, \text{Tr}) \cap L^1(\langle N, b \rangle, \text{Tr})$ for all $t > 0$. Applying the above identity, we have that the binormal states

$$\Phi_t : N \otimes_{\max} N^\circ \rightarrow \mathbb{C} \quad \Phi_t(x \otimes y^\circ) := \frac{1}{(\xi_\tau | T_t \xi_\tau)_{L^2(N, \tau)}} (i_\tau(y^*) | T_t(i_\tau(x)))_{L^2(N, \tau)},$$

well defined, by strong continuity of the semigroup, for t sufficiently close to zero, may be represented for $t > 0$ as $\Phi_t(x \otimes y^\circ) = \frac{1}{(\xi_\tau | T_t \xi_\tau)_{L^2(N, \tau)}} (T_t | x e_B y)_{L^2(\langle N, b \rangle, \text{Tr})}$.

By the identity above, Φ_t extends as a normal state on the von Neumann algebra generated by the left and right representations of N in $L^2(\langle N, B \rangle, \text{Tr})$. The N - N -correspondence \mathcal{H}_t generated by Φ_t is thus a sub-correspondence of a multiple of the N - N -correspondence $L^2(\langle N, B \rangle, \text{Tr})$. Since the semigroup $\{T_t : t > 0\}$ strongly converges to the identity operator on $L^2(N, \tau)$, we obtain that the trivial correspondence from N to N is weakly contained in the relative correspondence \mathcal{H}_B . \square

Example 5.7. (Minimal and maximal inclusions of a Dirichlet form) Let N be a von Neumann algebra and τ a normal, faithful, tracial state and let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(N, \tau)$ with associated self-adjoint operator $(L, D(L))$.

Assume that $\inf \sigma(L) = 0$ and that this is an eigenvalue (not necessarily discrete). The spectral projection P_0 onto the eigenspace corresponding to the Borel subset $\{0\} \subset [0, +\infty)$ can be represented as the strong limit $P_0 = \lim_{t \rightarrow +\infty} e^{-tL}$. Hence P_0 is a completely Markovian projection, so that there exists a von Neumann subalgebra $B_{\min} \subseteq N$ such that $P_0 = e_{B_{\min}}$. Obviously the associated Markovian semigroup is B_{\min} -bimodular and the Dirichlet form is B_{\min} -invariant.

Alternatively, one can consider the inclusion $B_{\max} \subset N$ where $B_{\max} := \{T_t : t > 0\}' \cap N$ is the relative commutant of the Markovian semigroup in N . Notice that by the Spectral Theorem $B_{\max} = \{T_t\}' \cap N$ for all $t > 0$. Obviously the associated Markovian semigroup is B_{\max} -bimodular and the Dirichlet form is B_{\max} -invariant.

Proposition 5.8. Let (N, τ) be a finite von Neumann algebra with faithful, normal trace. Let (\mathcal{E}, F) a Dirichlet form on $L^2(N, \tau)$ with generator $(L, D(L))$ having pure point spectrum made by distinct, isolated eigenvalues $\sigma(L) := \{\lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ and assume $\lambda_0 := \inf \sigma(L) = 0$.

Then (\mathcal{E}, F) has discrete spectrum relative to B_{\min} (resp. B_{\max}) if and only if each eigenspace $E_\lambda \subset L^2(N, \tau)$, $\lambda \in \sigma(L)$, has finite dimension $[E_\lambda : B_{\min}] < +\infty$ (resp. $[E_\lambda : B_{\max}] < +\infty$) relative to B_{\min} resp. B_{\max} .

Remark that the finite dimension $[E_\lambda : B_{\min}] < +\infty$ (resp. $[E_\lambda : B_{\max}] < +\infty$) relative to B_{\min} (resp. B_{\max}) is well defined for any eigenvalue $\lambda \in \sigma(L)$ because any eigenspace E_λ is obviously a left (and also right) B_{\min} -module (resp. B_{\max} -module).

Example 5.9. Let $H < \Gamma$ be an inclusion of countable, discrete groups and let $L(H) \subset L(\Gamma)$ be the inclusion of the finite von Neumann algebras generated by H and Γ , respectively. Their standard spaces coincide with $l^2(H)$ and $l^2(\Gamma)$ respectively and the projection $e_{L(H)}$ coincides with the projection from $l^2(\Gamma)$ onto its subspace $l^2(H)$.

Let $\ell : \Gamma \rightarrow [0, +\infty)$ be a c.n.d. function. The Dirichlet form $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ associated to ℓ (introduced in Section 3.3) is $L(H)$ -invariant if and only if ℓ vanishes on H or, equivalently, if ℓ is a right H -invariant function. In this situation we have:

- i) $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ is B_{\min} and B_{\max} -invariant;
- ii) $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to B_{\min} (resp. B_{\max}) if and only if each eigenspace has finite dimension $[E_\lambda : B_{\min}] < +\infty$ (resp. $[E_\lambda : B_{\max}] < +\infty$) relative to B_{\min} (resp. B_{\max}).

Proposition 5.10. Let Γ be a countable, discrete group and let $L(\Gamma)$ be its left von Neumann algebras. Let $\ell : \Gamma \rightarrow [0, +\infty)$ be a c.n.d. function and $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ the associated Dirichlet form. Denote by $H := \{s \in \Gamma : \ell(s) = 0\}$ the subgroup where ℓ vanishes. We then have

- i) $B_{\min} = B_{\max} = L(H)$;
- ii) If K is a subgroup of G , then $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ is $L(K)$ -invariant if and only if $K < H$. In this case:
 - ii.a) ℓ is $L(K)$ -biinvariant
 - ii.b) $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to $L(K) \subset L(\Gamma)$ if and only if the function

$$\ell_{G/K} : G/K \rightarrow [0, +\infty) \quad \ell_{G/K}(\tilde{s}) := \ell(s)$$

defined for $\tilde{s} = sK \in G/K$, is proper.

ii.c) If, for any $t > 0$, $\sum_{\tilde{s} \in G/K} e^{-t\ell_{G/K}(\tilde{s})} < +\infty$, then the inclusion $L(K) \subset L(G)$ is amenable.

Proof. i) Let $x = \sum_{t \in \Gamma} x(t)\lambda(t) \in B_{\max} = \{T_t : t > 0\}' \cap L(\Gamma)$. We have then $x\delta_e = x(I + L)^{-1}\delta_e = (I + L)^{-1}x\delta_e$, which implies $0 = L(x\delta_e) = L(\sum_{t \in \Gamma} x(t)\lambda(t)\delta_e) = L(\sum_{t \in \Gamma} x(t)\delta_t) = \sum_{t \in \Gamma} x(t)\ell(t)\delta_t$. So that $x(t)\ell(t) = 0$ for all $t \in \Gamma$ which in turn implies $x \in L(H) = B_{\min}$. The reverse inclusion is obvious.

ii) follows from the arguments of the example above. For ii) b) just notice that $\lambda(s)e_{L(K)}\lambda(s)^{-1}$ is the orthogonal projection P_{sK} onto the subspace $l^2(sH)$. Hence the eigenspace E_λ corresponding to the eigenvalue $\lambda \in \sigma(L)$ is given by $\bigoplus_{\tilde{s} \in G/K, \ell(s)=\lambda} l^2(sK)$. Hence, L will have discrete spectrum relative to K if and only if each of these sums is finite (i.e. for all λ) and the set of values of ℓ is discrete, i.e. $\ell^{-1}(\{\lambda\})/K$ is finite in G/K , i.e. $\ell_{G/K} : G/K \rightarrow [0, +\infty)$ is proper.

For ii.c), note that in the basic construction for $B = L(K) \subset N = L(G)$, $P_{sK} = \lambda(s)e_B\lambda(s)$ belongs to $\langle N, B \rangle$ and has trace 1. Hence $\text{Tr}(e^{-tL}) = \sum_{\tilde{s} \in G/K} e^{t\ell(s)}$ for all $t > 0$. \square

Remark 5.11.

i) If the Dirichlet form $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to $L(K)$ then

a) the function $\ell_{G/H} : G/H \rightarrow [0, +\infty)$ is proper and $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to $L(H)$;

b) $\ell_{G/K}$ being left K -invariant, hence constant on left K -cosets, and proper, left K -cosets in G/K must be finite sets. In other words, K is quasi-normal in G .

ii) On the other hand, if $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to $L(H)$ then the function $\ell_{G/K} : G/K \rightarrow [0, +\infty)$ will be constant onto the right H -coset in G/K . Thus $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ has discrete spectrum relative to $L(K)$ if and only if each right H -coset in G/K is a finite union of K -cosets, which happens if and only if K has finite index in H , i.e. when the homogeneous space H/K is finite.

6. A SPECTRAL APPROACH TO THE RELATIVE HAAGERUP PROPERTY

As already mentioned in the Introduction, in a recent work [CaSk], M. Caspers and A. Skalski characterized property (H) von Neumann algebras in terms of existence of a Dirichlet form with the discrete spectrum. In the spirit of the previous section, we extend their result to *relative property (H)*, as defined by S. Popa [Po1,2], for inclusions of von Neumann algebras, using a completely different approach.

Lemma 6.1. *Let (N, τ) be a von Neumann algebra endowed with a normal, faithful trace and let $\varphi : N \rightarrow N$ be a completely positive, normal contraction such that $\tau \circ \varphi \leq \tau$. Then*

i) *there exists a contraction $T_\varphi \in \mathcal{B}(L^2(N, \tau))$ such that*

$$T_\varphi(x\xi_\tau) = \varphi(x)\xi_\tau \quad x \in N;$$

ii) *there exists a completely positive, normal contraction $\varphi^* : N \rightarrow N$ such that*

$$T_{\varphi^*} = (T_\varphi)^*$$

or, more explicitly,

$$(\varphi^*(y)\xi_\tau | x\xi_\tau) = (y\xi_\tau | \varphi(x)\xi_\tau) \quad x, y \in N.$$

Proof. By the Cauchy-Schwarz inequality for τ , the Kadison-Schwarz inequality for φ and the hypothesis on the contraction of the trace τ by φ , for $x \in N$ one has first

$$\|\varphi(x)\xi_\tau\|_2 = \tau(\varphi(x)^*\varphi(x)) \leq \tau(\varphi(x^*x)) \leq \tau(x^*x) = \|x\xi_\tau\|_2$$

which proves i). Then, for $x, y \in N$, one has also

$$|\tau(y\varphi(x))| \leq \sqrt{\tau(y^*y)} \cdot \sqrt{\tau(\varphi(x)^*\varphi(x))} \leq \sqrt{\tau(y^*y)} \cdot \sqrt{\tau(\varphi(x^*x))} \leq \sqrt{\tau(y^*y)} \cdot \sqrt{\tau(x^*x)}.$$

This implies the existence of a normal map $\varphi^* : N \rightarrow L^2(N, \tau)$ characterized by

$$(\varphi^*(y)|x\xi_\tau) = (y\xi_\tau|\varphi(x)\xi_\tau) \quad x, y \in N.$$

This identity implies that φ^* maps the positive cone N_+ into the positive standard cone $L_+^2(N, \tau)$ and, by amplification, $\varphi^* \otimes I_{\mathbb{M}_n(\mathbb{C})}$ maps the positive cone $\mathbb{M}_n(N)_+$ into the standard positive cone $L_+^2(\mathbb{M}_n(N), \tau \otimes \text{tr})$. Moreover, for $x \in N$ one has

$$0 \leq \tau(\varphi^*(1_N)x) = (\tau \circ \varphi)(x) \leq \tau(x) = \|x\|_{L^1(N, \tau)}$$

which proves that $\varphi^*(1_N) \in N$ with $\|\varphi^*(1_N)\| \leq 1$. More generally, we have

$$0 \leq \varphi(y) \leq \varphi^*(1_N) \cdot \|y\|_N \leq \|y\|_N \cdot 1_N \quad y \in N_+.$$

By restriction, the map φ^* defines a completely positive, normal contraction of N . To end the argument, note that

$$\tau(\varphi^*(y)) = \tau(y \cdot \varphi(1_N)) \leq \|\varphi(1_N)\|_N \cdot \tau(y) \leq \tau(y) \quad y \in N_+.$$

□

Definition 6.2. ([Po1,2]) Let N be a finite von Neumann algebra and $B \subseteq N$ a von Neumann subalgebra. Then N is said to have *Property (H) relative to B* if there exist a normal, faithful tracial state τ on N and a net $\{\varphi_i : i \in I\}$ of normal completely positive, B -bimodular maps on N satisfying the conditions

- i) $\tau \circ \varphi_i \leq \tau$
- ii) $T_{\varphi_i} \in \mathcal{J}(\langle N, B \rangle)$
- iii) $\lim_{i \in I} \|x\xi_\tau - T_{\varphi_i}(x\xi_\tau)\|_2 = 0$ for all $x \in N$.

In this definition $\mathcal{J}(\langle N, B \rangle)$ is the compact ideal space, i.e. the norm closed ideal generated by projections with finite trace in $\langle N, B \rangle$ and T_{φ_i} is the operator defined in Lemma 6.1.

By a remark of S. Popa [Po2], the maps φ_i in the definition above can be chosen to be contractions. In the following we shall always assume this property for approximating nets of the identity map of a von Neumann algebra.

Theorem 6.3. *Let N be a finite von Neumann algebra with countably decomposable center and faithful tracial state τ . Let $B \subseteq N$ be a sub-von Neumann algebra such that $L^2(N, \tau)$, as B -module, admits a countable base. Then the following properties are equivalent*

- i) N has *Property (H) relative to B*
- ii) *there exists a B -invariant Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ with discrete spectrum relative to B .*

Proof. Assume that there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ with discrete spectrum relative to B . Hence, the associated generator $(L, D(L))$ has its resolvent in the compact ideal space: $(\lambda + L)^{-1} \in \mathcal{J}(\langle N, B \rangle)$. Then for all $\lambda > 0$, $S_\lambda := \lambda(\lambda + L)^{-1} \in \mathcal{J}(\langle N, B \rangle)$. Moreover, any S_λ is Markovian on $L^2(N, \tau)$ which implies that there exists a completely positive contraction $\varphi_\lambda : N \rightarrow N$ determined by $S_\lambda(x\xi_\tau) = \varphi_\lambda(x)\xi_\tau$ for $x \in N$. Since the S_λ are self-adjoint on $L^2(N, \tau)$, the φ_λ are symmetric with respect to the trace: $\tau(\varphi_\lambda(x)y) = \tau(x\varphi_\lambda(y))$ for all $x, y \in N$. This implies that

$$\tau(\varphi_\lambda(x)) = \tau(\varphi_\lambda(1_N)x) \leq \tau(x) \quad x \in N_+.$$

Last condition iii) in Definition 6.2 above comes from the strong continuity of the resolvent:

$$\lim_{\lambda \rightarrow +\infty} \|\xi - S_\lambda \xi\|_2 = 0 \quad \xi \in L^2(N, \tau).$$

The theorem is proved in the "if" direction. In the reverse direction, let us suppose that $B \subseteq N$ is an inclusion with relative property (H) and that $L^2(N, \tau)$ is separable as B -module. Let $\{\varphi_n : n \in \mathbb{N}\}$ be a sequence of normal, completely positive, B -bimodular contractions of N , satisfying the conditions of the definition above. By [Po2 Proposition 2.2] such a sequence always exists. Each φ_n extends by $T_n(x\xi_\tau) := \varphi_n(x)\xi_\tau$ to a B -bimodular contraction T_n of $L^2(N, \tau)$, which belongs to the compact ideal space $\mathcal{J}(\langle N, B \rangle)$. It is also completely positive with respect to the standard positive cone $L_+^2(N, \tau)$ and its matrix amplifications. It is easy to check that the maps φ_n^* of Lemma 6.1. have the same properties as the φ_n 's in definition above. Replacing each φ_n by $(\varphi_n + \varphi_n^*)/2$, we can suppose, without loss of generality, that the φ_n are symmetric with respect to τ so that the corresponding T_n are completely positive, self-adjoint contractions on $L^2(N, \tau)$.

Let $\{\xi_k \in L^2(N, \tau) : k \in \mathbb{N}\}$ be an orthonormal base for the left B -module $L^2(N, \tau)$. Recall that this means that $L^2(N, \tau) = \overline{\oplus_{k \in \mathbb{N}} B\xi_k}$.

For any $k \in \mathbb{N}$ one has $\lim_{n \rightarrow +\infty} T_n \xi_k = \xi_k$ and hence there exists $n_k \in \mathbb{N}$ such that

$$(\xi_j | (I - T_{n_k}) \xi_j) \leq 2^{-k}, \quad k \in \mathbb{N}, \quad j \in \{0, 1, \dots, k\}.$$

Let us consider the quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ defined by

$$\mathcal{E}[\xi] := \sum_{k \in \mathbb{N}} (\xi | (I - T_{n_k}) \xi) \quad \xi \in L^2(N, \tau).$$

The domain $\mathcal{F} \subseteq L^2(N, \tau)$ being understood as the subspace where the quadratic form is finite. Note first that \mathcal{E} is densely defined since $b\xi_j \in \mathcal{F}$ for all $b \in B$ and $j \in \mathbb{N}$. Noticing that each $\xi \mapsto (\xi | (I - T_{n_k}) \xi)$ is a bounded symmetric Dirichlet form on $L^2(N, \tau)$ we see that $(\mathcal{E}, \mathcal{F})$ is a lower semicontinuous, hence closed Dirichlet form on $L^2(N, \tau)$. The B -modularity of $(\mathcal{E}, \mathcal{F})$ being obvious what is left to prove is the relative discrete spectrum property.

@ The generator $(L, D(L))$ associated to $(\mathcal{E}, \mathcal{F})$ given by $L = \sum_{k \in \mathbb{N}} (I - T_{n_k})$, appears as the increasing limit $L = \lim_{m \uparrow +\infty} L_m$ of the bounded operators

$$L_m := \sum_{k=0}^m (I - T_{n_k}) = (m+1)I - \Theta_m$$

where $\Theta_m := \sum_{k=0}^m T_{n_k}$. The important fact is that Θ_m belongs to the compact ideal space $\mathcal{J}(\langle N, B \rangle)$.

Let q_m be the spectral projection of Θ_m corresponding to the interval $[0, (m+1)/2]$ and $p_m := I - q_m$ the spectral projection corresponding to the interval $((m+1)/2, m+1]$. On one hand we have $\tau(p_m) < +\infty$, which implies $(I + L)^{-1/2} p_m \in \mathcal{J}(\langle N, B \rangle)$. On the other hand by spectral calculus we have

$$q_m(I + L)^{-1} q_m \leq q_m(I + L_m)^{-1} q_m \leq \frac{2}{m+1} q_m$$

which implies

$$\|q_m(I + L)^{-1/2}\| \leq \sqrt{\frac{2}{m+1}}.$$

Finally, we have $(I + L)^{-1/2} = \lim_{m \rightarrow +\infty} (I + L)^{-1/2} p_m$ for the uniform norm, which implies that $(I + L)^{-1/2}$ and $(I + L)^{-1}$ are in $\mathcal{J}(\langle N, B \rangle)$. \square

7. RELATIVE PROPERTY (H) FOR INCLUSIONS OF DISCRETE GROUPS AND CONDITIONALLY NEGATIVE DEFINITE FUNCTIONS

In this section we extend a well known characterization of groups with Property (H) in terms of the existence of a proper c.n.d. function, to inclusions of discrete groups $H < G$. We denote by λ and ρ the left and right regular representation of G in the Hilbert space $l^2(G)$. If $(\delta_t)_{t \in G}$ is the canonical orthonormal base of $l^2(G)$, then

$$\lambda(s)\delta_t = \delta_{ts} \quad \rho(s)\delta_t = \delta_{ts^{-1}} \quad s, t \in G.$$

The associated inclusion of von Neumann algebras is

$$B = L(H) = \lambda(H)'' \subset \lambda(G)'' = L(G) = N.$$

$L(G)$ has the canonical finite trace $\tau : L(G) \rightarrow \mathbb{C}$ determined by $\tau(\lambda(s)) = \delta_e(s)$ for $s \in G$. The standard space $L^2(L(G), \tau)$ identifies canonically with $l^2(G)$ through the unitary map determined by $\lambda(s)\xi_\tau \rightarrow \delta_s$ for $s \in G$. The projection e_B of the basic construction is the orthogonal projection from $l^2(G)$ onto its subspace $l^2(H)$ given by the multiplication operator by the characteristic function χ_H of the subset H of G .

Lemma 7.1. (*Basic construction for group inclusions*).

- i) The basic construction $\langle N, B \rangle$ is the commutant $\rho(H)'$ of the right regular representation restricted to H .
- ii) For $T \in \langle N, B \rangle$, the map $\varphi_T : G \rightarrow \mathbb{C}$ given by $s \mapsto (\delta_s, T\delta_s)$ is right H -invariant:

$$\varphi_T(sh) = \varphi_T(s) \quad s \in G, h \in H.$$

- iii) The canonical trace on $\langle N, B \rangle$ is given by the formula

$$\text{Tr}(T) = \sum_{\tilde{s} \in G/H} \varphi_T(s) = \sum_{\tilde{s} \in G/H} (\delta_s, T\delta_s) \quad T \in \langle N, B \rangle$$

where it is understood that the map $\tilde{s} \mapsto s$ indicates a section of the projection $G \rightarrow G/H$.

Proof. i) On one side, we have $\langle N, B \rangle = (JBH)'$. On the other side, it is an elementary fact that $J\lambda(h)J = \rho(h)$, $h \in H$. ii) For $h \in H$ and $T \in \mathcal{B}(l^2(G))$ commuting with $\rho(h)$, we compute

$$\varphi_T(sh) = (\rho(h)^{-1}\delta_s, T\rho(h)^{-1}\delta_s) = (\delta_s, \rho(h)T\rho(h)^{-1}\delta_s) = \varphi_T(s) \quad s \in G, h \in H.$$

- iii) The formula

$$\varphi(T) = \sum_{\tilde{s} \in G/H} \varphi_T(s) = \sum_{\tilde{s} \in G/H} (\delta_s, T\delta_s) \in [0, +\infty] \quad T \in \langle N, B \rangle_+ = \rho(H)'_+$$

defines a normal faithful weight on the von Neumann algebra $\langle N, B \rangle$.

For $x = \sum_{s \in G} x(s) \lambda(s) \in L(G)$, let us compute, using an arbitrary section $\sigma : G/H \rightarrow G$ of the homogeneous space G/H :

$$\begin{aligned}
\varphi(x^* e_B x) &= \sum_{s, t \in G} \sum_{\gamma \in G/H} \overline{x(t)} x(s) (\delta_{\sigma(\gamma)}, \lambda(t)^* e_B \lambda(s) \delta_{\sigma(\gamma)}) \\
&= \sum_{s, t \in G} \sum_{\gamma \in G/H} \overline{x(t)} x(s) (\delta_{t\sigma(\gamma)}, e_B \delta_{s\sigma(\gamma)}) \\
&= \sum_{s, t \in G} \sum_{\gamma \in G/H} \overline{x(t)} x(s) (\delta_{t\sigma(\gamma)}, \chi_H \delta_{s\sigma(\gamma)}) \\
&= \sum_{\gamma \in G/H} \sum_{s, t \in G, s\sigma(\gamma) \in H, t=s} \overline{x(t)} x(s) \\
&= \sum_{\gamma \in G/H} \sum_{s \in H\sigma(\gamma)^{-1}} |x(s)|^2 \\
&= \sum_{s \in G} |x(s)|^2 = \tau(x^* x) = \text{Tr}(x^* e_B x).
\end{aligned}$$

Which proves that the normal weight φ is semifinite and that it coincides with the canonical trace Tr (check that its density with respect to Tr is the identity operator). \square

Theorem 7.2. *Let G be a countable discrete group and $H < G$ a subgroup. Then the inclusion of von Neumann algebras $L(H) \subset L(G)$ has the relative Property (H) if and only if there exists a conditionally negative type function $\ell : G \rightarrow [0, +\infty)$ such that*

- i) $\ell|_H = 0$
- ii) ℓ is proper on G/H .

Proof. Let us suppose that i) and ii) are satisfied. Then for any $t > 0$, $e^{-t\ell}$ is a positive type function equal to 1 on H and it induces a normal, completely positive, trace preserving contraction ϕ_t on the von Neumann algebra $L(G)$, characterized by

$$\phi_t(\lambda_u) = e^{-t\ell(u)} \lambda_u \quad u \in G.$$

Let (π, H_π) be the orthogonal representation and $c : G \rightarrow H_\pi$ the 1-cocycle associated to the c.n.d. function ℓ . If $\ell(u) = 0$ then $0 = \ell(u) = \|c(u)\|_{H_\pi}^2$ so that $c(u) = 0$ for all $u \in H$. Then, $c(vu) = c(v) + \pi(v)c(u) = c(v)$ for all $v \in G$ and $u \in H$ and $\ell(vu) = \ell(v)$, $v \in G$, $u \in H$. Finally, as $\ell(v^{-1}) = \ell(v)$ for all $v \in G$, one also has $\ell(uv) = \ell(v)$ for all $v \in G$ and $u \in H$.

Consequently we shall have

$$\phi_t(\lambda_v \lambda_u) = e^{-t\ell(vu)} \lambda_v \lambda_u = e^{-t\ell(v)} \lambda_v \lambda_u = \phi_t(\lambda_v) \lambda_u \quad v \in G, u \in H$$

and

$$\phi_t(ab) = \phi_t(a)b \quad a \in L(G), b \in L(H).$$

Similarly

$$\phi_t(ba) = b\phi_t(a) \quad a \in L(G), b \in L(H).$$

This proves that ϕ_t is a $L(H)$ -bimodular map for all $t > 0$.

The self-adjoint operator T_t on $l^2(G)$, induced by ϕ_t on $L(G)$, is just the multiplication operator by the function $e^{-t\ell}$. Its spectrum coincides with the set of values $e^{-t\lambda}$ where λ runs in the range $\ell(G)$ which is, by assumption ii), a discrete subset of $[0, +\infty)$.

The eigenspace E_λ , corresponding to the eigenvalue $e^{-t\lambda}$, is the set of functions in $l^2(G)$ supported by $S_\lambda := \{s \in G : \ell(s) = \lambda\}$. Again by assumption ii), S_λ is a finite union of right

H -cosets: $S_\lambda = \bigcup_{i=1}^k u_i H$ for some $u_1, \dots, u_k \in G$. The corresponding spectral projection P_λ is then the sum of projections on those cosets:

$$P_\lambda = \sum_{i=1}^k \text{multiplication by } \chi_{u_i H}.$$

Since, for $B := L(H) \subset L(G) = N$, the projection e_B is just the multiplication operator by the characteristic function χ_H of the subgroup, the projection $\lambda_u e_B \lambda_u^{-1}$, for $u \in G$, is the multiplication operator by the function χ_{uH} . The spectral projection P_λ onto E_λ is then a finite sum of projections of trace one (for the trace Tr), and its trace is equal to the number of right H -cosets in S_λ :

$$\text{Tr}(P_\lambda) = \text{Tr}\left(\sum_{i=1}^k \lambda_{u_i} e_B \lambda_{u_i}^{-1}\right) = \sum_{i=1}^k \text{Tr}(\lambda_{u_i} e_B \lambda_{u_i}^{-1}) = \sum_{i=1}^k \text{Tr}(e_B) = \sum_{i=1}^k 1 = k.$$

This proves that T_t belong to $\mathcal{J}(\langle N, B \rangle)$, which ends the proof in the direct sense.

Conversely, let us suppose that the inclusion $L(H) \subset L(G)$ has the relative property (H) . By Theorem 6.3, there exists an $L(H)$ -invariant symmetric Dirichlet form with generator L and discrete spectrum relative to the subalgebra $B = L(H)$.

Let us observe that, for $\varepsilon > 0$, the resolvent maps $(I + \varepsilon L)^{-1}$ are a completely positive, normal contractions of the von Neumann algebra $N = L(G)$ and that the map

$$\omega_\varepsilon : G \rightarrow [0, +\infty) \quad \omega_\varepsilon(s) = (\delta_s, (I + \varepsilon L)^{-1} \delta_s) = \tau(\lambda(s)^*(I + \varepsilon L)^{-1}(\lambda(s)))$$

is positive definite on G and H -right invariant. Moreover, by the weak*-continuity of the resolvent, one has

$$\lim_{\varepsilon \rightarrow 0} \omega_\varepsilon(s) = 1 \quad s \in G.$$

We claim that ω_ε vanishes at infinity on the quotient space G/H . This holds true because $(I + \varepsilon L)^{-1}$ being in the compact ideal $\mathcal{J}(\langle N, B \rangle)$, it will be a uniform limit of trace class operators $T_n \in L^1(\langle N, B \rangle, Tr)$ and that, for such T_n , the function $s \mapsto (\delta_s, T_n \delta_s)$ is summable on G/H (by Lemma 7.1 iii) thus vanishing at infinity.

Let $(F_k)_{k \geq 1}$ be an increasing family of finite subsets of G such that $\bigcup_k F_k = G$. For any k , let us choose $\varepsilon_k > 0$ such that

$$0 \leq 1 - \omega_{\varepsilon_k}(s) \leq 2^{-k} \quad s \in F_k$$

and consider the function $\ell : G \rightarrow [0, +\infty)$ defined by

$$\ell(s) = \sum_{k=1}^{\infty} (1 - \omega_{\varepsilon_k}(s)) \quad s \in G.$$

By the choice of ε_k , the series converges for any s in any F_k and thus for any $s \in G$ so that ℓ is well defined on G . The function ℓ is c.n.d. as a sum of c.n.d. functions and a right H -invariant function as a sum of right- H -invariant functions. It can thus be considered as a function on G/H .

Moreover, as the functions ω_ε vanish at infinity on G/H , the set $\Gamma_k = \{\tilde{s} \in G/H \mid \omega_{\varepsilon_k}(s) \geq 1/2\}$ is finite and for $\tilde{s} \notin \bigcup_{k=1}^N \Gamma_k$, one has $\ell(s) \geq N/2$. Hence, for any $N \in \mathbb{N}$, the set $\{\tilde{s} \in G/H \mid \ell(s) \leq N/2\}$ is finite, which proves that ℓ is proper on G/H . \square

Corollary 7.3. *If $H < G$ and $L(H) \subset L(G)$ has relative property (H) , then H is a quasi normal subgroup: each orbit of the left action of H on the right-cosets space G/H is finite.*

Proof. The c.n.d. function ℓ constructed in Theorem 7.2 is left H -invariant, as it clearly satisfies $\ell(s^{-1}) = \ell(s)$ for $s \in G$. It is thus constant on the orbits of the left action of H on the right cosets space G/H . A subset of G/H on which the proper function ℓ is constant must be finite. \square

Corollary 7.4. *If $H < G$ is normal, then the inclusion $L(H) \subset L(G)$ has relative property (H) if and only if the quotient group G/H has the Haagerup property.*

Example 7.5. Let us consider the free group \mathbb{F}_2 with generators $a, b \in \mathbb{F}_2$ and the abelian subgroup $H := \{a^k : k \in \mathbb{Z}\}$ isomorphic to the additive group \mathbb{Z} of integer numbers.

H is not quasi-normal in \mathbb{F}_2 , so that the inclusion $H < G$ has not the relative property (H) , though both G and H have property (H) .

More generally, for countable discrete groups G_1, G_2 , the inclusion $G_1 < G_1 * G_2$ is not quasi-normal as soon as G_1 is infinite and G_2 has at least two elements. Hence, this inclusion has not the relative property H .

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